

SYMMETRIC GROUP CHARACTERS AS SYMMETRIC FUNCTIONS

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ABSTRACT. We show that the irreducible characters of the symmetric group are symmetric polynomials evaluated at the eigenvalues of permutation matrices. In fact, these characters can be realized as symmetric functions that form a non-homogeneous basis for the ring of symmetric functions. We call this basis the *irreducible character basis*. Further, the structure coefficients for the (outer) product of these functions are the stable Kronecker coefficients.

The induced trivial characters also give rise to a non-homogeneous basis of symmetric functions. We introduce the irreducible character basis by defining it in terms of the *induced trivial character basis*. In addition, we show that the irreducible character basis is closely related to character polynomials and we obtain some of the change of basis coefficients by making this connection explicit. Other change of basis coefficients come from a representation theoretic connection with the partition algebra, and still others are derived by developing combinatorial expressions.

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1. INTRODUCTION

The ring of symmetric functions and the representation theory of both the symmetric group, S_k , and the general linear group, GL_n , are deeply connected. It is a well-known fact that the ring of symmetric functions Sym is the “universal” character ring of GL_n . In fact, irreducible GL_n characters are obtained as evaluations of Schur functions, s_λ , at the eigenvalues of the elements in GL_n . In addition, operations on symmetric functions correspond to operations on GL_n representations. For example, tensor products of irreducible GL_n representations correspond to the product of Schur functions.

On the other hand, the irreducible characters of S_k , χ^λ , occur as change of basis coefficients when we write the power symmetric functions, p_μ , in terms of the Schur functions,

$$(1) \quad p_\mu = \sum_{\lambda \vdash k} \chi^\lambda(\mu) s_\lambda$$

where $\chi^\lambda(\mu)$ is the value of χ^λ at the conjugacy class indexed by μ . Furthermore, induction and restriction of S_k representations corresponds to the product and coproduct of Schur functions.

$$s_\lambda s_\mu = \sum_{\nu} c_{\lambda\mu}^\nu s_\nu \text{ and } \Delta(s_\nu) = \sum_{\lambda, \mu} c_{\lambda\mu}^\nu s_\lambda \otimes s_\mu$$

where $c_{\lambda\mu}^\nu$ are the Littlewood-Richardson coefficients.

The facts stated above are all consequences of the Schur-Weyl duality between GL_n and S_k . This duality arises when we consider the diagonal action of GL_n and the permutation action of S_k on $(\mathbb{C}^n)^{\otimes k}$. That is, if $V = \mathbb{C}^n$, then GL_n acts diagonally on $V^{\otimes k}$, then

$$A \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_k) = Av_1 \otimes Av_2 \otimes \cdots \otimes Av_k, \quad \text{for } A \in GL_n$$

and S_k acts on $V^{\otimes k}$ by permuting tensor coordinates. These two actions commute. This implies that $V^{\otimes k}$ is a $S_k \times GL_n$ -module and it decomposes as follows:

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k} S^\lambda \otimes W^\lambda,$$

where the S^λ are the irreducible representations of S_k , W^λ are irreducible representations of GL_n and λ ranges over all partitions of k with at most n parts. Then Equation (1) follows from this by computing the character of $V^{\otimes k}$.

As a consequence of the duality we obtain that tensoring representations of GL_n corresponds to restricting representations of S_k , that is,

$$[W^\lambda \otimes W^\mu : W^\nu] = c_{\lambda, \mu}^\nu = [S^\nu \downarrow_{S_r \times S_t}^{S_{r+t}} : S^\lambda \boxtimes S^\mu],$$

where $[W : M]$ denotes the multiplicity of M in W and \boxtimes denotes the outer product.

Similar results have been obtained for other subgroups of GL_n . For example, Schur-Weyl duality between orthogonal and symplectic groups and Brauer algebras have been used to compute characters for Brauer algebras [Ram]. In addition, Koike and Terada [KT] developed symmetric function expressions for characters of orthogonal and symplectic groups.

In the early 1990s Martin [Ma1, Ma2, Ma3, Ma4] and Jones [Jo] introduced the *partition algebra*, $P_k(n)$. Jones showed that if we identify S_n with the permutation matrices in GL_n and act diagonally on $V^{\otimes k}$, then the centralizer algebra we obtain is $P_k(n)$. This means that S_n and $P_k(n)$ are Schur-Weyl duals of each other. Similar as in the classical case, we also have that $V^{\otimes k}$ decomposes as a $P_k(n) \times S_n$ -module as follows

$$V^{\otimes k} = \bigoplus_{\lambda \vdash n} L^\lambda \otimes S^\lambda$$

where L^λ are irreducible representation of $P_k(n)$ and the sum runs over all partitions of n such that $\lambda_2 + \lambda_3 + \dots + \lambda_{\ell(\lambda)} \leq k$.

This duality has been used by Halverson [Hal] to compute the characters of the partition algebra by describing a Murnaghan-Nakayama rule for the characters of $P_k(n)$. In [BDO], Bowman, De Visscher and Orellana used this duality to show that

$$[S^\lambda \otimes S^\mu : S^\nu] = \bar{g}_{\lambda, \mu}^\nu = [L^\nu \downarrow_{P_r \times P_t}^{P_{r+t}} : L^\lambda \boxtimes L^\mu],$$

where $\bar{g}_{\lambda, \mu}^\nu$ are the stable (or reduced) Kronecker coefficients and n is sufficiently large.

The main motivation for this paper is the introduction of a new non-homogeneous basis for the ring of symmetric functions. This new basis, $\{\tilde{s}_\lambda\}$, are the universal characters of the symmetric group. We call this new basis the *irreducible character basis*. The irreducible character basis $\{\tilde{s}_\lambda\}$ plays the same role for the symmetric group as the Schur functions $\{s_\lambda\}$ do for the irreducible characters of GL_n . In addition, we also introduce the *induced trivial character basis*, $\{\tilde{h}_\lambda\}$. We show that the irreducible character of the symmetric group indexed by $(n - |\lambda|, \lambda)$ can be recovered when we evaluate \tilde{s}_λ at roots of unity. In addition, we show that

$$\tilde{s}_\lambda \tilde{s}_\mu = \sum_\nu \bar{g}_{\lambda, \mu}^\nu \tilde{s}_\nu$$

That is, the (outer) product of the \tilde{s}_λ 's have structure coefficients given by the stable Kronecker coefficients [BOR1, Murg2, Murg3]. We also have in analogy with (1) the following relation with the partition algebra characters:

$$p_\mu = \sum_\lambda \chi_{P_k(n)}^\lambda(d_\mu) \tilde{s}_\lambda$$

where $\chi_{P_k(n)}^\lambda(d_\mu)$ denotes the irreducible character of $P_k(n)$ indexed by λ and evaluated at an element d_μ which is analogous to a conjugacy class representative (see [Hal] for further details).

We point out that our new bases provide a unifying mantle for many different objects connected with Kronecker coefficients. For example, in Section 6 we give a Murnaghan-Nakayama rule for multiplying \tilde{s}_λ by the power symmetric function p_k (for a positive integer k). This allows us to compute the characters of the partition algebra and we obtain the equivalent result of that in [Hal]. Also, a link to character polynomials is provided in Section 5 by following the presentation of character polynomials found in [GG]. Further, our analysis of these new bases have led to the introduction of new combinatorial objects that will be useful for describing the algebraic expressions involving symmetric group characters.

We hope that the bases introduced in this paper will lead to progress on many hard problems on the representation theory of the symmetric group. For example, we envision that our bases will be useful to make progress on the Kronecker problem, the plethysm problem and the restriction of irreducible GL_n representations to S_n .

1.1. Summary of results. We now describe the main results of our paper.

- (1) The evaluation of symmetric functions at roots of unity give character values for the restriction of GL_n -representations to the symmetric group. In Section 8, we find combinatorial interpretations for values of the specializations of h_λ (see Theorem 2) and e_λ (see Corollary 51) at roots of unity. These are the GL_n characters of the tensor products of the symmetric and alternating representations.
- (2) The introduction of non-homogeneous bases $\{\tilde{h}_\lambda\}$ and $\{\tilde{s}_\lambda\}$ for the ring of symmetric functions is in Sections 3. The \tilde{s}_λ are symmetric functions such that when we evaluate at the eigenvalues of permutation matrices we get the irreducible characters of the symmetric group. Similarly, the \tilde{h}_λ evaluate to the characters of the induced trivial module for the symmetric group when we specialize to the eigenvalues of permutation matrices.
- (3) In Section 3 through 6, we study the transition coefficients between our new bases and the classical bases for the symmetric functions. If $\{u_\lambda\}_\lambda$ and $\{v_\lambda\}_\lambda$ are two bases, denote by $u_\lambda \rightarrow v_\mu$ the transition coefficient of v_μ when u_λ is written in the v -basis. In Equation (4) we give $h_\lambda \rightarrow \tilde{h}_\mu$. The transitions $\tilde{s}_\lambda \rightarrow \tilde{h}_\mu$ and $\tilde{h}_\lambda \rightarrow \tilde{s}_\mu$ are related by Kostka coefficients in Equation (7) and (8). This allows us to give in Theorem 5 the transition $h_\lambda \rightarrow \tilde{s}_\mu$ in terms of multiset valued tableaux and in Theorem 8 the transition $e_\lambda \rightarrow \tilde{s}_\mu$ in terms of set valued tableaux. Using a connection with character polynomials we are able to give expansions for $\tilde{h}_\lambda \rightarrow p_\mu$ (Equation (35)) and $\tilde{s}_\lambda \rightarrow p_\mu$ (Equations (27), (37), (38)). Finally, in Equation (41) we describe $p_\lambda \rightarrow \tilde{s}_\mu$.
- (4) In Section 5, we give an explicit description of the relationship between the \tilde{s} -basis and the character polynomials. We show that the function \tilde{s}_λ can be obtained as a specialization of the character polynomials when we replace the variables in these polynomials by linear combinations of power symmetric functions. For details, see Proposition 11.
- (5) In Section 6, we show that the irreducible characters of the partition algebra occur as change of basis coefficients when we expand the power symmetric basis in terms of the \tilde{s} -basis. This leads to a symmetric function version of the Murnagham-Nakayama rule (see Theorem 19).
- (6) The structure coefficients for the \tilde{s} -basis are the stable (or reduced) Kronecker coefficients. In Proposition 21 we give a combinatorial interpretation for the structure coefficients of the \tilde{h} -basis. This yields a new combinatorial interpretation for the Kronecker product of two homogeneous symmetric functions. A different combinatorial interpretation for this was found by Garsia and Remmel [GR].

2. NOTATION AND PRELIMINARIES

The combinatorial objects that arise in our work are the classical building blocks: set, multiset, partition, set partition, multiset partition, composition, weak composition, tableau, words, etc. In this section we remind the reader about the definitions of these objects as well as establish notation and usual conventions that we will use in this paper.

For non-negative integers n and ℓ , a partition of size n and length ℓ is a sequence of positive integers, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$, such that $\lambda_i \geq \lambda_{i+1}$ for $1 \leq i < \ell$ and $\lambda_1 + \lambda_2 + \dots + \lambda_\ell = n$. The size of the partition is denoted $|\lambda| = n$ and the length of the partition is denoted $\ell(\lambda) = \ell$. We will often use the shorthand notation $\lambda \vdash n$ to indicate that λ is a partition of n . The symbols λ and μ will be reserved exclusively for partitions. Let $m_i(\lambda)$ represent the number of times that i appears in the partition λ . Sometimes it will be convenient to represent our partitions in exponential notation where $m_i = m_i(\lambda)$ and $\lambda = (1^{m_1} 2^{m_2} \dots k^{m_k})$. With this notation the number of permutations with cycle structure $\lambda \vdash n$ is $\frac{n!}{z_\lambda}$ where

$$(2) \quad z_\lambda = \prod_{i=1}^{\lambda_1} m_i(\lambda)! i^{m_i(\lambda)} .$$

The most common operation we use is that of adding a part of size n to the beginning of a partition. This is denoted (n, λ) . If $n < \lambda_1$, this sequence will no longer be an integer partition and we will have to interpret the object appropriately.

The Young diagram (or cells) of a partition λ are the set of points $\{(i, j) : 1 \leq i \leq \lambda_j, 1 \leq j \leq \ell(\lambda)\}$. We will represent these cells as stacks of boxes in the first quadrant (following ‘French notation’ for a partition). A tableau is a mapping from the set of cells to a set of labels and a tableau will be represented by filling the boxes of the diagram of a partition with the labels. In our case, we will encounter tableaux where only a subset of the cells are mapped to a label. A tableau T is column strict if $T(i, j) \leq T(i+1, j)$ and $T(i, j) < T(i, j+1)$ for all the filled cells of the tableau. The total content of a tableau is the multiset obtained with the total number of occurrences of each number.

A multiset (sets with repeated elements) will be denoted by $\{\!\{b_1, b_2, \dots, b_r\}\!\}$. Multisets will also be represented by exponential notation so that $\{\!\{1^{a_1}, 2^{a_2}, \dots, \ell^{a_\ell}\}\!\}$ represents the multiset where the value i occurs a_i times.

A set partition of a set S is a set of subsets $\{S_1, S_2, \dots, S_\ell\}$ with $S_i \subseteq S$ for $1 \leq i \leq \ell$, $S_i \cap S_j = \emptyset$ for $1 \leq i < j \leq \ell$ and $S_1 \cup S_2 \cup \dots \cup S_\ell = S$. A multiset partition $\pi = \{\!\{S_1, S_2, \dots, S_\ell\}\!\}$ of a multiset S is a similar construction to a set partition, but now S_i may be a multiset, and it is possible that two multisets S_i and S_j have non-empty intersection (and may even be equal). The length of a multiset partition is denoted by $\ell(\pi) = \ell$. We will use the notation $\pi \Vdash S$ to indicate that π is a multiset partition of the multiset S .

We will use $\tilde{m}(\pi)$ to represent the partition of $\ell(\pi)$ consisting of the the multiplicities of the multisets which occur in π (e.g. $\tilde{m}(\{\!\{1, 1, 2\}\!\}, \{\!\{1, 1, 2\}\!\}, \{\!\{1, 3\}\!\}) = (2, 1)$ because $\{\!\{1, 1, 2\}\!\}$ occurs 2 times and $\{\!\{1, 3\}\!\}$ occurs 1 time).

For non-negative integers n and ℓ , a composition of size n is an ordered sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$ such that $\alpha_1 + \alpha_2 + \dots + \alpha_\ell = n$. A weak composition is such a sequence with the condition that $\alpha_i \geq 0$ (zeros are allowed). To indicate that α is a composition of n we will use the notation $\alpha \models n$ and to indicate that α is a weak composition of n we will use the notation $\alpha \models_w n$. For both compositions and weak compositions, $\ell(\alpha) := \ell$.

Remark 1. Multiset partitions of a multiset are isomorphic to objects called vector partitions which have previously been used to index symmetric functions in multiple sets of variables [Comtet, MacMahon, Rosas]. Since multiset partitions are more amenable to tableaux we have used this alternate combinatorial description.

2.1. The ring of symmetric functions. For some modern references on this subject see for example [Mac, Sagan, Stanley, Lascoux]. The ring of symmetric functions will be denoted $Sym = \mathbb{Q}[p_1, p_2, p_3, \dots]$. The p_k are power sum generators and they will be thought of as functions which can be evaluated at values when appropriate by making the substitution $p_k \rightarrow x_1^k + x_2^k + \dots + x_n^k$ but they are used algebraically in this ring without reference to their variables.

The fundamental bases of Sym (each indexed by the set of partitions λ) are *power sum* $\{p_\lambda\}_\lambda$, *homogeneous/complete* $\{h_\lambda\}_\lambda$, *elementary* $\{e_\lambda\}_\lambda$, and *Schur* $\{s_\lambda\}_\lambda$. The Hall inner product is defined by declaring that the power sum basis is orthogonal, i.e., $\left\langle \frac{p_\lambda}{z_\lambda}, p_\mu \right\rangle =$

$\delta_{\lambda=\mu}$, where we use the notation $\delta_A = \begin{cases} 1 & \text{if } A \text{ is true} \\ 0 & \text{if } A \text{ is false} \end{cases}$. Under this inner product the

Schur functions are orthonormal, $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda=\mu}$. We use this scalar product to represent values of coefficients by taking scalar products with dual bases. We will also refer to the irreducible character of the symmetric group indexed by the partition λ and evaluated at a permutation of cycle structure μ as the coefficient $\langle s_\lambda, p_\mu \rangle = \chi^\lambda(\mu)$. For $k > 0$, define

$$\Xi_k := 1, e^{2\pi i/k}, e^{4\pi i/k}, \dots, e^{2(k-1)\pi i/k}$$

as a symbol representing the eigenvalues of a permutation matrix of a k -cycle. Then for any partition μ , let

$$\Xi_\mu := \Xi_{\mu_1}, \Xi_{\mu_2}, \dots, \Xi_{\mu_{\ell(\mu)}}$$

be the multiset of eigenvalues of a permutation matrix with cycle structure μ . We will evaluate symmetric functions at these eigenvalues. The notation $f[\Xi_\mu]$ represents taking the element $f \in Sym$ and replacing p_k in f with $x_1^k + x_2^k + \dots + x_{|\mu|}^k$ and then replacing the variables x_i with the values in Ξ_μ .

3. SYMMETRIC GROUP CHARACTER BASES OF THE SYMMETRIC FUNCTIONS

We have relegated a large part of the necessary buildup of these symmetric function bases to two appendices in Section 8 and 9. It is not that the results in those sections are not interesting, it is just that their presentation detracts from the main goal of this paper, which is to introduce the *induced trivial character basis*, $\{\tilde{h}_\lambda\}$, and the *irreducible*

character basis, $\{\tilde{s}_\lambda\}$. The evaluations of these families of symmetric functions at roots of unity (the eigenvalues of a permutation matrix) will be the values of characters.

We provide a proof of the following result in Section 8.

Theorem 2. *For all partitions λ and μ , let $H_{\lambda\mu} := \langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \rangle$. We have the evaluation,*

$$(3) \quad h_\lambda[\Xi_\mu] = \sum_{\pi \vdash \{\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}\}} H_{\tilde{m}(\pi), \mu}.$$

Note that $H_{\lambda\mu} = 0$ if $|\mu| - |\lambda| < 0$.

$$(4) \quad h_\lambda = \sum_{\pi \vdash \{\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}\}} \tilde{h}_{\tilde{m}(\pi)}.$$

This is a recursive definition for calculating this basis directly since there is precisely one multiset partition of $\{\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}\}$ such that $\tilde{m}(\pi)$ is of size $|\lambda|$, hence

$$(5) \quad \tilde{h}_\lambda = h_\lambda - \sum_{\substack{\pi \vdash \{\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}\} \\ \tilde{m}(\pi) \neq \lambda}} \tilde{h}_{\tilde{m}(\pi)}.$$

Now by equation (3) and an induction argument, we can conclude that for all partitions μ ,

$$(6) \quad \tilde{h}_\lambda[\Xi_\mu] = H_{\lambda\mu} = \langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \rangle$$

and this is the value of the character of the trivial module which is induced from $S_{|\mu|-|\lambda|} \times S_{\lambda_1} \times S_{\lambda_2} \times \dots \times S_{\lambda_{\ell(\lambda)}}$ to the full symmetric group $S_{|\mu|}$. We call these symmetric functions ‘characters’ because we can think of them as functions that can be evaluated on the eigenvalues of permutation matrices and these evaluations are equal to the character values of symmetric group representations. In particular, the \tilde{h}_λ is a symmetric function whose evaluations at the eigenvalues of a permutation matrix are the value of the character of this induced trivial module. We therefore call the symmetric functions $\{\tilde{h}_\lambda\}$, as a basis of the symmetric functions, the *induced trivial character basis*.

Example 3. To compute a small example: $\tilde{h}_1 = h_1$. Since $\{\{1\}\}$ and $\{\{1, 1\}\}$ are the two multiset partitions of $\{\{1, 1\}\}$, then $h_2 = \tilde{h}_1 + \tilde{h}_2$. Therefore $\tilde{h}_2 = h_2 - h_1$.

Next we define the symmetric functions \tilde{s}_λ by using the Kostka coefficients (the change of basis coefficients between the complete, h_μ , and Schur bases, s_λ , are denoted $K_{\lambda\mu}$) as the change of basis coefficients with \tilde{h}_λ basis. Choose an $n \geq 2|\mu|$ then we take as definition of the elements \tilde{s}_λ the basis that satisfies

$$(7) \quad \tilde{h}_\mu = \sum_{|\lambda| \leq |\mu|} K_{(n-|\lambda|, \lambda)(n-|\mu|, \mu)} \tilde{s}_\lambda.$$

Alternatively, the coefficient of \tilde{s}_λ in \tilde{h}_μ is equal to $\sum_\gamma K_{\gamma\mu}$ where the sum is over partitions γ such that γ/λ is a horizontal strip (at most one cell in each row) of size $|\mu| - |\lambda|$. This also implies that we can express \tilde{s}_λ in terms

$$(8) \quad \tilde{s}_\lambda = \sum_{|\mu| \leq |\lambda|} K_{(n-|\lambda|, \lambda)(n-|\mu|, \mu)}^{-1} \tilde{h}_\mu$$

where n is any positive integer greater than or equal to $2|\lambda|$ and $K_{\lambda\mu}^{-1}$ are the inverse Kostka coefficients (the change of basis coefficients between the Schur basis and the complete symmetric function basis). There is a combinatorial interpretation for the Kostka coefficients $K_{\lambda\mu}$ as the number of column strict tableaux of shape λ and content μ . Using this interpretation we can show that $K_{(n-|\lambda|, \lambda)(n-|\mu|, \mu)}$ is independent of the value of n as long as n is sufficiently large. That is, if n is sufficiently large, then for any $m \geq n$, $K_{(m-|\lambda|, \lambda)(m-|\mu|, \mu)} = K_{(n-|\lambda|, \lambda)(n-|\mu|, \mu)}$.

If n is smaller than $|\lambda| - \lambda_1$, then the change of basis coefficients are the same as those between the complete symmetric functions and a Schur function indexed by a composition $\alpha = (|\mu| - |\lambda|, \lambda)$, namely the expression representing the Jacobi-Trudi matrix

$$(9) \quad s_\alpha = \det [h_{\alpha_i + i - j}]_{1 \leq i, j \leq \ell(\alpha) + 1}.$$

We find then that \tilde{s}_λ are the (unique) symmetric functions of inhomogeneous degree $|\lambda|$ that evaluate to the characters of the symmetric group, that is

$$(10) \quad \begin{aligned} \tilde{s}_\lambda[\Xi_\gamma] &= \sum_{|\mu| \leq |\lambda|} K_{(n-|\lambda|, \lambda)(n-|\mu|, \mu)}^{-1} \tilde{h}_\mu[\Xi_\gamma] \\ &= \sum_{|\mu| \leq |\lambda|} K_{(n-|\lambda|, \lambda)(n-|\mu|, \mu)}^{-1} \langle h_{n-|\mu|} h_\mu, p_\gamma \rangle \\ &= \langle s_{(n-|\lambda|, \lambda)}, p_\gamma \rangle. \end{aligned}$$

If $n \geq |\lambda| + \lambda_1$ then this last expression is equal to the value of the irreducible character $\chi^{(n-|\lambda|, \lambda)}(\gamma)$.

We call the basis \tilde{s}_λ the characters of the irreducible representations of the symmetric group when the symmetric group is realized as permutation matrices. They are characters in the same way that the Schur functions are the characters of the irreducible representations of the general linear group. We therefore name the basis \tilde{s}_λ the *irreducible character basis*.

Because these functions are characters, the structure coefficients of the product of this basis are the *reduced Kronecker coefficients*.

Recall that the Kronecker product is the bilinear product on symmetric functions defined on the power sum basis by

$$(11) \quad \frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \delta_{\lambda=\mu} \frac{p_\lambda}{z_\lambda}.$$

The symbol $\delta_{\lambda=\mu}$ is the Kronecker delta function that is equal to 1 if $\lambda = \mu$ and 0 otherwise. Since $\left\langle \frac{p_\lambda}{z_\lambda}, p_\mu \right\rangle = \delta_{\lambda=\mu}$, we can verify the trivial calculation

$$(12) \quad \left\langle \frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu}, p_\gamma \right\rangle = \left\langle \frac{p_\lambda}{z_\lambda}, p_\gamma \right\rangle \left\langle \frac{p_\mu}{z_\mu}, p_\gamma \right\rangle = \delta_{\lambda=\gamma} \delta_{\mu=\gamma} .$$

Since our product and scalar product are bilinear, we have for any symmetric functions f and g ,

$$(13) \quad \langle f * g, p_\gamma \rangle = \langle f, p_\gamma \rangle \langle g, p_\gamma \rangle$$

This implies that since $\tilde{s}_\lambda[\Xi_\gamma] = \langle s_{(|\gamma|-|\lambda|, \lambda)}, p_\gamma \rangle$ and $\tilde{h}_\lambda[\Xi_\gamma] = \langle h_{(|\gamma|-|\lambda|, \lambda)}, p_\gamma \rangle$ that the structure coefficients for the irreducible character and induced trivial character functions are the same as the Kronecker coefficients.

By work of Murnaghan [Murg2, Murg3], there exists coefficients $\bar{d}_{\lambda\mu}^\gamma$ and $\bar{g}_{\lambda\mu}^\gamma$ with the property that

$$(14) \quad s_{(n-|\lambda|, \lambda)} * s_{(n-|\mu|, \mu)} = \sum_{\gamma} \bar{g}_{\lambda\mu}^\gamma s_{(n-|\gamma|, \gamma)}$$

and

$$(15) \quad h_{(n-|\lambda|, \lambda)} * h_{(n-|\mu|, \mu)} = \sum_{\gamma} \bar{d}_{\lambda\mu}^\gamma h_{(n-|\gamma|, \gamma)}$$

for all $n \geq 0$. The $\bar{g}_{\lambda\mu}^\gamma$ are usually referred to as ‘reduced’ or ‘stable’ Kronecker coefficients (see for example [Aiz, BOR, Kly]).

When we compute the product of these functions, the structure coefficients are given by the stable Kronecker coefficients.

Theorem 4. *For partitions λ and μ ,*

$$(16) \quad \tilde{h}_\lambda \tilde{h}_\mu = \sum_{|\nu| \leq |\lambda| + |\mu|} \bar{d}_{\lambda\mu}^\nu \tilde{h}_\nu \text{ and } \tilde{s}_\lambda \tilde{s}_\mu = \sum_{|\nu| \leq |\lambda| + |\mu|} \bar{g}_{\lambda\mu}^\nu \tilde{s}_\nu .$$

Proof. We begin by evaluating the product of these functions at the eigenvalues of a permutation matrix.

$$\begin{aligned} \tilde{h}_\lambda[\Xi_\gamma] \tilde{h}_\mu[\Xi_\gamma] &= \langle h_{(|\gamma|-|\lambda|, \lambda)}, p_\gamma \rangle \langle h_{(|\gamma|-|\mu|, \mu)}, p_\gamma \rangle \\ &= \langle h_{(|\gamma|-|\lambda|, \lambda)} * h_{(|\gamma|-|\mu|, \mu)}, p_\gamma \rangle \\ &= \sum_{|\nu| \leq |\lambda| + |\mu|} \bar{d}_{\lambda\mu}^\nu \langle h_{(|\gamma|-|\nu|, \nu)}, p_\gamma \rangle \\ &= \sum_{|\nu| \leq |\lambda| + |\mu|} \bar{d}_{\lambda\mu}^\nu \tilde{h}_\nu[\Xi_\gamma] . \end{aligned}$$

and a similar calculation shows,

$$\tilde{s}_\lambda[\Xi_\gamma]\tilde{s}_\mu[\Xi_\gamma] = \sum_{|\nu| \leq |\lambda| + |\mu|} \tilde{g}_{\lambda\mu}^\nu \tilde{s}_\nu[\Xi_\gamma] .$$

Since these expressions are an identity for all γ of sufficiently large size, we conclude by Corollary 56 that the theorem holds. \square

4. THE IRREDUCIBLE CHARACTER EXPANSION OF A COMPLETE AND ELEMENTARY SYMMETRIC FUNCTION

The combinatorial interpretation of the coefficients for the irreducible character expansion of a complete symmetric function comes from combining the notion of multiset partition of a multiset and column strict tableau. To work with column strict tableaux on sets or multisets we need to establish a total order on these objects. We remark that, with few restrictions, we can do this with almost any total order and so we will use lexicographic if we read the entries of the multiset in increasing order. This may mean that the tableaux we work with will have evaluation which is not a partition, but this is typical with column strict tableaux.

If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)})$ is a partition, then use the notation $\bar{\lambda} = (\lambda_2, \lambda_3, \dots, \lambda_{\ell(\lambda)})$ to represent the partition with the first part removed. For a tableau T , let $\text{shape}(T)$ denote the partition of the outer shape of the tableau.

Theorem 5. *For a partition μ ,*

$$(17) \quad h_\mu = \sum_T \tilde{s}_{\text{shape}(T)}$$

the sum is over all skew-shape column strict tableaux of shape $\lambda/(\lambda_2)$ for some partition λ where the cells are filled with non-empty multisets of labels such that the total content of the tableau is $\{\!\{1^{\mu_1}, 2^{\mu_2}, \dots, \ell^{\mu_\ell}\}\!\}$.

Proof. From equation (4), we know the expansion of h_μ in terms of multiset partitions of a multiset and by (7) we know the expansion of $\tilde{h}_{\tilde{m}(\pi)}$ in the \tilde{s}_λ basis terms of column strict tableaux. Combining these two expansions we have that for an n sufficiently large,

$$(18) \quad h_\mu = \sum_{\pi \Vdash \{\!\{1^{\mu_1}, 2^{\mu_2}, \dots, \ell^{\mu_\ell}\}\!\}} \sum_{\lambda \vdash n} K_{\lambda, (n-\ell(\pi), \tilde{m}(\pi))} \tilde{s}_{\bar{\lambda}} .$$

Now we note that for every multiset partition π and column strict tableaux of shape λ and content given by the partition $(n - \ell(\pi), \tilde{m}(\pi))$ we can create a skew-shaped tableau whose entries are multisets by replacing the $n - \ell(\pi)$ labels with a 1 with a blank so that it is of skew shape $\lambda/(n - \ell(\pi))$ and the other labels by their corresponding multiset in π . The value of n needs to be chosen to be sufficiently large so that it is larger than the size of the first part of $\bar{\lambda}$.

To explain why this is equal to the description stated in the theorem where there are precisely λ_2 blank cells in the first row, we note that $K_{\lambda, (n-\ell(\pi), \tilde{m}(\pi))} = K_{(n' - |\bar{\lambda}|, \bar{\lambda}), (n' - \ell(\pi), \tilde{m}(\pi))}$ as long as $n' - \ell(\pi) \geq \lambda_2$. This is because there is a bijection between these two sets

of tableaux by inserting or deleting 1s in the first row of each tableau in the set. In particular, we may choose $n' - \ell(\pi) = \lambda_2$ and the description of the tableaux are precisely those that are column strict of skew of shape $(n' - |\bar{\lambda}|, \bar{\lambda})/(\lambda_2)$ and whose entries are the multisets in π . \square

Example 6. Consider the following 20 column strict tableaux whose entries are multisets with total content of the tableau $\{1^2, 2\}$.

$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 2 \\ \hline 1 & 1 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & \\ \hline & & 2 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & \\ \hline & & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array}$	$\begin{array}{ c c } \hline 2 \\ \hline 1 & \\ \hline \end{array}$	$\begin{array}{ c c } \hline 2 \\ \hline 1 & \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & & \\ \hline & 1 & 2 \\ \hline \end{array}$
$\begin{array}{ c c c } \hline 2 & & \\ \hline & 1 & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & \\ \hline & 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 2 & \\ \hline & 1 & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 2 \\ \hline & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 1 \\ \hline & 2 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c } \hline 1 & 2 \\ \hline \end{array}$

Theorem 5 states then that

$$(19) \quad h_{21} = 4\tilde{s}_{()} + 7\tilde{s}_1 + 3\tilde{s}_{11} + 4\tilde{s}_2 + \tilde{s}_{21} + \tilde{s}_3 .$$

Example 7. Let us compute the decomposition of $V^{\otimes 4}$ where $V = \mathcal{L}\{x_1, x_2, x_3, \dots, x_n\}$ as an S_n module with the diagonal action. The module V has character equal to $\tilde{h}_1 = h_1$. Therefore to compute the decomposition of this character into S_n irreducibles we are looking for the expansion of h_{1^4} into the irreducible character basis.

Using Sage [sage, sage-combinat] we compute that it is

$$(20) \quad \begin{aligned} h_{1^4} = & 15\tilde{s}_{()} + 37\tilde{s}_1 + 31\tilde{s}_{11} + 10\tilde{s}_{111} + \tilde{s}_{1111} + 31\tilde{s}_2 \\ & + 20\tilde{s}_{21} + 3\tilde{s}_{211} + 2\tilde{s}_{22} + 10\tilde{s}_3 + 3\tilde{s}_{31} + \tilde{s}_4 \end{aligned}$$

If $n \geq 6$ then the multiplicity of the irreducible $(n - 3, 3)$ will be 10. The combinatorial interpretation of this value is the number of column strict tableaux with entries that are multisets (or in this case sets) of $\{1, 2, 3, 4\}$ of skew-shape $(4, 3)/(3)$ or $(3, 3)/(3)$. Those tableaux are

<table><tr><td>1</td><td>2</td><td>3</td><td></td></tr><tr><td></td><td></td><td></td><td>4</td></tr></table>	1	2	3					4	<table><tr><td>1</td><td>2</td><td>4</td><td></td></tr><tr><td></td><td></td><td></td><td>3</td></tr></table>	1	2	4					3	<table><tr><td>1</td><td>3</td><td>4</td><td></td></tr><tr><td></td><td></td><td></td><td>2</td></tr></table>	1	3	4					2	<table><tr><td>2</td><td>3</td><td>4</td><td></td></tr><tr><td></td><td></td><td></td><td>1</td></tr></table>	2	3	4					1																		
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What is interesting about this example is that the usual combinatorial interpretation for the repeated Kronecker product $(\chi^{(n-1,1)} + \chi^{(n)})^{*k}$ is stated in other places in the literature in terms of oscillating tableaux [CG, Sund]. Thus, this special case of Theorem 5 gives a new combinatorial description of the multiplicities in terms of set valued tableaux.

The irreducible character expansion of an elementary symmetric function is similar. We again assume that there is a total order on the sets that appear in set partitions and create tableaux to keep track of the terms in the symmetric function expansion. Let $shape(T)$ be a partition representing the shape of a tableau T and we again use the overline notation on a partition to represent the partition with the first part removed, $\bar{\lambda} = (\lambda_2, \lambda_3, \dots, \lambda_{\ell_\lambda})$.

Theorem 8. *For a partition μ ,*

$$(21) \quad e_\mu = \sum_T \tilde{s}_{\text{shape}(T)}$$

where the sum is over tableaux that are skew shape $\lambda/(\lambda_2)$ for some partition λ and that are weakly increasing in rows and columns with non-empty sets as labels of the tableaux (not multisets so that no repeated values in the sets allowed) such that the content of the tableau is $\{\{1^{\mu_1}, 2^{\mu_2}, \dots, \ell^{\mu_\ell}\}\}$. A set is allowed to appear multiple times in the same column if and only if the set has an odd number of entries. A set is allowed to appear multiple times in the same row if and only if the set has an even number of entries.

Proof. From Corollary 51 we know that

$$(22) \quad e_\lambda[\Xi_\mu] = \sum_{\pi \vdash \{\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}\}} \langle h_n h_{\tilde{m}_e(\pi)} e_{\tilde{m}_o(\pi)}, p_\mu \rangle .$$

For the rest of this proof, we assume that the reader is familiar with the Pieri rules which state $h_r s_\lambda$ is the sum of terms s_μ where the Young diagram for μ differs from the Young diagram of λ by adding cells that may occur in the same row, but not in the same column of the diagram. Similarly, $e_r s_\lambda$ is the sum of terms s_μ where the Young diagram for μ differs from the Young diagram of λ by adding cells that may occur in the same column, but not in the same row of the diagram.

Now order the occurrences of the generators in the product $h_n h_{\tilde{m}_e(\pi)} e_{\tilde{m}_o(\pi)}$ so that they are in the same order as the total order that is chosen for the sets that appear in the set partitions. Using a tableau to keep track of the terms in the resulting Schur expansion of the product we will have that

$$(23) \quad h_n h_{\tilde{m}_e(\pi)} e_{\tilde{m}_o(\pi)} = \sum_T s_{\text{shape}(T)}$$

where the sum is over tableaux that have n blank cells in the first row and sets as labels in the rest of the tableau. Because we multiply by a generator h_r if a set with an even number of entries occurs r times, then those sets with an even number of elements can appear multiple times in the same row, but not in the same column of the tableau. Similarly, because we multiply by a generator e_r if a set with an odd number of entries occurs r times, then those sets with an odd number of elements can appear multiple times in the same column, but not in the same row of the tableau.

As was discussed in the proof of Theorem 5, if the shape of the tableau is of skew shape $\lambda/(n)$ with $n \geq \lambda_2$, there are the same number of tableaux of skew-shape $(\lambda_2 + |\bar{\lambda}|, \bar{\lambda})/(\lambda_2)$ because there is a bijection by deleting blank cells in the first row.

Therefore we have

$$(24) \quad e_\lambda[\Xi_\mu] = \sum_{\pi \vdash \{\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}\}} \sum_T \langle s_{\text{shape}(T)}, p_\mu \rangle = \sum_{\pi \vdash \{\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}\}} \sum_T \tilde{s}_{\text{shape}(T)}[\Xi_\mu]$$

where the sum is over those tableau described in the statement of the proposition. Our proposition now follows from Corollary 56. \square

Example 9. To begin with a small example, consider the expansion of e_{21} . The following 11 tableaux follow the rules outlined in Theorem 8.

$\begin{array}{ c c } \hline 1 & \\ \hline 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c } \hline 2 \\ \hline 1 \\ \hline 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & \\ \hline 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & 2 & \\ \hline & & 1 \\ \hline \end{array}$	$\begin{array}{ c c c } \hline 1 & & 2 \\ \hline & 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 2 \\ \hline 1 & \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 1 & 2 \\ \hline & & \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 2 \\ \hline & 1 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & \\ \hline & 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 1 & 2 \\ \hline & & \\ \hline \end{array}$
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Theorem 8 then states that

$$(25) \quad e_{21} = \tilde{s}_{()} + 3\tilde{s}_1 + 2\tilde{s}_2 + 3\tilde{s}_{11} + \tilde{s}_{21} + \tilde{s}_{111}$$

Example 10. A slightly larger example is the expansion of e_{33} . If we use Sage [sage, sage-combinat] to determine the expansion we see that

$$\begin{aligned} e_{33} = & 2\tilde{s}_{()} + 4\tilde{s}_1 + 4\tilde{s}_{11} + 4\tilde{s}_{111} + 4\tilde{s}_{1111} + 3\tilde{s}_{11111} + \tilde{s}_{111111} + 6\tilde{s}_2 \\ & + 8\tilde{s}_{21} + 7\tilde{s}_{211} + 4\tilde{s}_{2111} + \tilde{s}_{21111} + 5\tilde{s}_{22} + 4\tilde{s}_{221} + \tilde{s}_{2211} + \tilde{s}_{222} \\ & + 5\tilde{s}_3 + 4\tilde{s}_{31} + \tilde{s}_{311} + \tilde{s}_{32} + \tilde{s}_4 \end{aligned}$$

Hence there are 71 tableaux in total satisfying the conditions of Theorem 8 that have a total content $\{1^3, 2^3\}$. Listing all 71 tableaux is perhaps not a clear example so let's consider just the coefficient of \tilde{s}_{21} . If we order the sets $\{1\} < \{1, 2\} < \{2\}$ then to explain the coefficient of \tilde{s}_{21} then we consider the following 8 tableaux which are of skew-shape either $(4, 2, 1)/(2)$ or $(3, 2, 1)/(2)$.

$\begin{array}{ c c c c } \hline 1 & & & \\ \hline 1 & 2 & & \\ \hline & & 1 & 2 & 2 \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 1 & 2 & & \\ \hline 1 & 2 & & \\ \hline & & 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 2 & & & \\ \hline 1 & 1 & 2 & \\ \hline & & 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 2 & & & \\ \hline 1 & 2 & & \\ \hline & & 1 & 1 & 2 \\ \hline \end{array}$
$\begin{array}{ c c c c } \hline 2 & & & \\ \hline 1 & 2 & 1 & 2 \\ \hline & & & 1 \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 1 & 2 & & \\ \hline 1 & 2 & & \\ \hline & & 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 2 & & & \\ \hline 1 & 1 & 2 & \\ \hline & & 1 & 2 \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 1 & 2 & & \\ \hline 1 & 1 & 2 & \\ \hline & & & 2 \\ \hline \end{array}$

5. CHARACTER POLYNOMIALS AND THE IRREDUCIBLE CHARACTER BASIS

Character polynomials were first used by Murnaghan [Murg]. Much later, Specht [Sp] gave determinantal formulas and expressions in terms of binomial coefficients for these polynomials. They are treated as an example in Macdonald's book [Mac, ex. I.7.13 and I.7.14]. More recently, Garsia and Goupil [GG] gave an umbral formula for computing them. We will show in this section that character polynomials are a transformation of character symmetric functions and this will allow us to give an expression for character symmetric functions in the power sum basis.

Following the notation of [GG], a character polynomial is a multivariate polynomial $q_\lambda(x_1, x_2, x_3, \dots)$ in the variables x_i such that for specific integer values $x_i = m_i \in \mathbb{Z}$,

$$(26) \quad q_\lambda(m_1, m_2, m_3, \dots) = \chi^{(n-|\lambda|, \lambda)}(1^{m_1} 2^{m_2} 3^{m_3} \dots)$$

where $n = \sum_{i \geq 1} i m_i$. As a consequence of Lemma 55 and Proposition 54 we have the following relationship between the character polynomials $q_\lambda(x_1, x_2, x_3, \dots)$ and character basis \tilde{s}_λ .

Proposition 11. For $n \geq 0$ and $\lambda \vdash n$,

$$q_\lambda(x_1, x_2, x_3, \dots) = \tilde{s}_\lambda \Big|_{p_k \rightarrow \sum_{d|k} dx_d}$$

and

$$\tilde{s}_\lambda = q_\lambda(x_1, x_2, x_3, \dots) \Big|_{x_k \rightarrow \frac{1}{k} \sum_{d|k} \mu(k/d) p_d}$$

where $\Big|_{a_i \rightarrow b_i}$ means that we are replacing a_i with the expression b_i .

In [GG], the character polynomials are computed algorithmically. If we make an additional substitution, i.e., x_k by $\frac{1}{k} \sum_{d|k} \mu(k/d) p_d$, in their algorithm, then we obtain \tilde{s}_λ using the following steps.

- (1) Expand the Schur function s_λ in the power sums basis $s_\lambda = \sum_\gamma \frac{\chi^\lambda(\gamma)}{z_\gamma} p_\gamma$.
- (2) Replace each power sum p_i by $ix_i - 1$.
- (3) Expand each product $\prod_i (ix_i - 1)^{a_i}$ as a sum $\sum_g c_g \prod_i x_i^{g_i}$.
- (4) Replace each $x_k^{g_k}$ by $(x_k)_{g_k} = x_k(x_k - 1) \cdots (x_k - g_k + 1)$.
- (5) Replace each x_k by $\frac{1}{k} \sum_{d|k} \mu(k/d) p_d$.

Example 12. To compute \tilde{s}_3 we follow the steps to obtain:

- (1) $s_3 = \frac{1}{6}(p_1^3 + 3p_2 p_1 + 2p_3)$
- (2) $\frac{1}{6}(p_1^3 + 3p_2 p_1 + 2p_3) \rightarrow \frac{1}{6}((x_1 - 1)^3 + 3(2x_2 - 1)(x_1 - 1) + 2(3x_3 - 1))$
- (3) $\frac{1}{6}((x_1 - 1)^3 + 3(2x_2 - 1)(x_1 - 1) + 2(3x_3 - 1)) = \frac{1}{6}x_1^3 - \frac{1}{2}x_1^2 + x_1 x_2 - x_2 + x_3$
- (4) $q_3 = \frac{1}{6}(x_1)_3 - \frac{1}{2}(x_1)_2 + x_1 x_2 - x_2 + x_3$
- (5) $\tilde{s}_3 = \frac{1}{6}(p_1)_3 - \frac{1}{2}(p_1)_2 + p_1 \frac{p_2 - p_1}{2} - \frac{p_2 - p_1}{2} + \frac{p_3 - p_1}{3}$

As an important consequence, we derive a power sum expansion of the irreducible character basis by following the algorithm stated above.

Theorem 13. For $n \geq 0$ and $\lambda \vdash n$,

$$(27) \quad \tilde{s}_\lambda = \sum_{\gamma \vdash n} \chi^\lambda(\gamma) \frac{\mathbf{p}_\gamma}{z_\gamma}$$

where

$$(28) \quad \mathbf{p}_{i^r} = \sum_{k=0}^r (-1)^{r-k} i^k \binom{r}{k} \left(\frac{1}{i} \sum_{d|i} \mu(i/d) p_d \right)_k \quad \text{and} \quad \mathbf{p}_\gamma := \prod_{i \geq 1} \mathbf{p}_{i^{m_i(\gamma)}} ,$$

and $(x)_k$ denotes the k -th falling factorial.

Proof. The proof of this proposition is exactly the steps outlined in the result of [GG] and then add the additional step of replacing x_k with $\frac{1}{k} \sum_{d|k} \mu(k/d) p_d$. The Schur function has a power sum expansion given by

$$(29) \quad s_\lambda = \sum_{\gamma \vdash |\lambda|} \chi^\lambda(\gamma) \frac{p_\gamma}{z_\gamma} = \sum_{\gamma \vdash |\lambda|} \chi^\lambda(\gamma) \frac{1}{z_\gamma} \prod_{i=1}^{\ell(\gamma)} (p_i)^{m_i(\gamma)} .$$

Then in the next step we replace p_i with $ix_i - 1$ and expand the expression. The part of the expression $(p_i)^r$ becomes

$$(30) \quad (p_i)^r \Big|_{p_i \rightarrow ix_i - 1} = (ix_i - 1)^r = \sum_{k=0}^r (-1)^{r-k} i^k \binom{r}{k} x_i^k .$$

In the last step the [GG] algorithm replaces x_i^k with $(x_i)_k$ and that is the expression for the character polynomial. To recover the irreducible character function \tilde{s}_λ , we use Equation (88) and replace x_k with $\frac{1}{k} \sum_{d|k} \mu(k/d) p_d$. Replacing x_i^k in with $\left(\frac{1}{i} \sum_{d|i} \mu(i/d) p_d \right)_k$ means that $(p_i)^r$ will be replaced by the expression in Equation (28).

It follows that the composition of these steps changes s_λ to \tilde{s}_λ and the power sum expansion of the Schur function to the right hand side of Equation (27). \square

We also present a similar formula for the expansion of the induced trivial character basis in the power sum basis. To do this we introduce a basis which acts like an indicator function for evaluation at the roots of unity. Define

$$(31) \quad \bar{\mathbf{p}}_i^r = i^r \left(\frac{1}{i} \sum_{d|i} \mu(i/d) p_d \right)_r \quad \text{and} \quad \bar{\mathbf{p}}_\gamma := \prod_{i \geq 1} \bar{\mathbf{p}}_{i^{m_i(\gamma)}} .$$

Lemma 14. *For partitions γ and μ such that $|\mu| < |\gamma|$, then $\bar{\mathbf{p}}_\gamma[\Xi_\mu] = 0$. Moreover, if $|\mu| = |\gamma|$, then*

$$(32) \quad \bar{\mathbf{p}}_\gamma[\Xi_\mu] = z_\gamma \delta_{\gamma=\mu} .$$

Proof. Since $p_d[\Xi_\mu] = \sum_{d'|d} d' m_{d'}(\mu)$ and

$$(33) \quad \frac{1}{i} \sum_{d|i} \mu(i/d) \left(\sum_{d'|d} d' m_{d'}(\mu) \right) = m_i(\mu) ,$$

then plugging into (31) we see

$$(34) \quad \bar{\mathbf{p}}_{i^r}[\Xi_\mu] = i^r (m_i(\mu))_r .$$

and hence $\bar{\mathbf{p}}_\gamma[\Xi_\mu] = \prod_{i \geq 1} i^{m_i(\gamma)} (m_i(\mu))_{m_i(\gamma)}$.

Now if $|\mu| < |\gamma|$ or $|\mu| = |\gamma|$ and $\mu \neq \gamma$, then there exists at least one i such that $m_i(\mu) < m_i(\gamma)$ and for that value i , $(m_i(\mu))_{m_i(\gamma)} = 0$ and hence $\bar{\mathbf{p}}_\gamma[\Xi_\mu] = 0$.

If $\mu = \gamma$, then $\bar{\mathbf{p}}_\gamma[\Xi_\gamma] = \prod_{i \geq 1} i^{m_i(\gamma)} (m_i(\gamma))_{m_i(\gamma)} = \prod_{i \geq 1} i^{m_i(\gamma)} m_i(\gamma)! = z_\gamma$. \square

This basis can then be used to be a formula for the induced trivial character basis.

Proposition 15. *For $n \geq 0$ and $\lambda \vdash n$,*

$$(35) \quad \tilde{h}_\lambda = \sum_{\gamma \vdash n} \langle h_\lambda, p_\gamma \rangle \frac{\bar{\mathbf{p}}_\gamma}{z_\gamma} .$$

Proof. For any partition μ such that $|\mu| < |\lambda|$, then $\tilde{h}_\lambda[\Xi_\mu] = \langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \rangle = 0$ because $|\mu| - |\lambda| < 0$. In addition, by Lemma 14, $\sum_{\gamma \vdash |\lambda|} \langle h_\lambda, p_\gamma \rangle \frac{\bar{\mathbf{p}}_\gamma[\Xi_\mu]}{z_\gamma} = 0$. If $|\mu| = |\lambda|$, then $\langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \rangle = \langle h_\lambda, p_\mu \rangle$ and

$$(36) \quad \sum_{\gamma \vdash |\lambda|} \langle h_\lambda, p_\gamma \rangle \frac{\bar{\mathbf{p}}_\gamma[\Xi_\mu]}{z_\gamma} = \langle h_\lambda, p_\mu \rangle = \tilde{h}_\lambda[\Xi_\mu] .$$

We can conclude since we have equality for all evaluations at Ξ_μ for $|\mu| \leq |\lambda|$, then by Proposition 54 that Equation (35) holds at the level of symmetric functions. \square

Remark 16. After finding this formula for \tilde{h}_λ in terms of the elements $\bar{\mathbf{p}}_\gamma$ we noticed that a similar expression appears in Macdonald's book [Mac] on page 121. He defines polynomials for each partition ρ in variables a_1, a_2, \dots as $\binom{a}{\rho} := \prod_{r \geq 1} \binom{a_r}{m_r(\rho)}$. We noticed that if $a_i = \frac{1}{i} \sum_{d|i} \mu(i/d) p_d$ then $\binom{a}{\rho} = \frac{\bar{\mathbf{p}}_\rho}{z_\rho}$. The interested reader can translate Equation (4) from Example 14 on page 123 for the character polynomial to conclude that

$$(37) \quad \tilde{s}_\lambda = \sum_{\sigma, \rho} (-1)^{\ell(\sigma)} \langle s_\lambda, p_\rho p_\sigma \rangle \frac{\bar{\mathbf{p}}_\rho}{z_\sigma z_\rho}$$

summed over all partitions ρ and σ such that $|\rho| + |\sigma| = |\lambda|$. We can also translate Equation (5) from the same example on page 124 to show that

$$(38) \quad \tilde{s}_\lambda = \sum_{\mu} (-1)^{|\lambda|-|\mu|} \sum_{\gamma \vdash |\mu|} \langle s_\mu, p_\gamma \rangle \frac{\bar{\mathbf{p}}_\gamma}{z_\gamma}$$

where the outer sum is over partitions μ such that λ/μ is a vertical strip (no more than one cell in each row of the skew partition).

6. THE PARTITION ALGEBRA AND A MURNAGHAN-NAKAYAMA RULE

The partition algebra was independently defined in the work of Martin [Ma1, Ma2, Ma3, Ma4] and Jones [Jo]. Jones showed that the partition algebra is the centralizer algebra of the diagonal action of the symmetric group on tensor space. In other words he described a Schur-Weyl duality between the symmetric group and the partition algebra. Later, Halverson [Hal] described the analogues of the Frobenius formula and the Murnaghan-Nakayama rule to compute the characters of the partition algebra. In this section we describe a connection between the partition algebra characters and our irreducible character basis.

If V is the permutation representation of the symmetric group S_r , then the centralizer of the diagonal action on $V^{\otimes n}$ depends on the two parameters n and r and is denoted by $P_n(r)$. The irreducible characters are indexed by partitions $(r - |\lambda|, \lambda)$, where λ is a partition of size less than or equal to n . Halverson described conjugacy class analogues and denoted the representatives of this classes by d_μ , where μ is a partition of size less than or equal to n . Using these notations, the irreducible partition algebra character values are denoted by $\chi_{P_n(r)}^{(r-|\lambda|, \lambda)}(d_\mu)$.

Corollary 4.2.3 of [Hal] states the following properties of the partition algebra characters.

Corollary 17. *If $|\lambda| \leq n$ and μ is a composition of size less than or equal to n , then*

- (1) $\chi_{P_n(r)}^{(r-|\lambda|, \lambda)}(d_\mu) = \begin{cases} r^{n-|\mu|} \chi_{P_{|\mu|}(r)}^{(r-|\lambda|, \lambda)}(d_\mu) & \text{if } |\mu| \geq |\lambda|, \\ 0 & \text{if } |\mu| < |\lambda| \end{cases}$
- (2) $|\mu| = |\lambda| = n$, then $\chi_{P_n(r)}^{(r-|\lambda|, \lambda)}(d_\mu) = \chi_{S_n}^\lambda(\mu)$
- (3) if $r \geq 2n$ and $|\mu| = n$, then $\chi_{P_n(r)}^{(r-|\lambda|, \lambda)}(d_\mu)$ is independent of r .

For a positive integer n and $\mu \vdash n$, the usual Frobenius formula for the symmetric functions is a consequence of the classical Schur-Weyl duality between GL_n and GL_r , it states

$$(39) \quad p_\mu = \sum_{\lambda \vdash n} \chi_{S_n}^\lambda(\mu) s_\lambda$$

where s_λ (as symmetric functions) are the irreducible GL_r characters.

If we restrict the diagonal action of GL_r to the symmetric group, S_r , realized by the permutation matrices, we obtain the Schur-Weyl duality between the Symmetric group S_r and the partition algebra $P_n(r)$. A decomposition of $V^{\otimes n}$ as a $(P_n(r), S_n)$ -module into irreducibles yields the analogue of the Frobenius formula for $P_n(r)$ and the symmetric group. See equation (3.2.1) and Theorem 3.2.2 of [Hal] where we assume that $r \geq 2|\mu|$, and $\gamma \vdash r$,

$$(40) \quad p_\mu[\Xi_\gamma] = \sum_{|\lambda| \leq |\mu|} \chi_{P_{|\mu|}(r)}^{(r-|\lambda|, \lambda)}(d_\mu) \tilde{s}_\lambda[\Xi_\gamma] .$$

Since this identity hold for all r greater than a fixed value and all partitions $\gamma \vdash r$, then by Corollary 56, this expression is a symmetric function identity and we have

$$(41) \quad p_\mu = \sum_{|\lambda| \leq |\mu|} \chi_{P_{|\mu|}(r)}^{(r-|\lambda|, \lambda)}(d_\mu) \tilde{s}_\lambda .$$

Our next result is a statement which is equivalent to the Murnaghan-Nakayama rule for the computation of the irreducible symmetric group characters. To state how this relation appears in the irreducible character basis, we first introduce a little notation.

For a partitions λ and ν such that $\nu \subseteq \lambda$ and $r \geq \max(|\lambda| + \lambda_1, |\nu| + \nu_1)$, let $\lambda //_{\nu} = (r - |\lambda|, \lambda) / (r - |\nu|, \nu)$ represent a skew partition, i.e. a subcollection of cells in the Young diagram of $(r - |\lambda|, \lambda)$. We will say that $\lambda //_{\nu}$ is a k border strip (abbreviated $\lambda //_{\nu} \in B_k$) if it consists of k cells which are connected and do not contain a 2×2 sub-configuration of cells. We refer to the elements in B_k as k border strips. Let $ht(\lambda //_{\nu})$ equal to the number of rows occupied by the skew partition minus 1.

As the size of r doesn't really matter as long as it is sufficiently large, we will assume that r is much larger than the partitions that we are working with and drop the r from the $//_r$ notation.

Lemma 18. For $n, k > 0$ and $\lambda \vdash n$, let $\mu \vdash 2n + k$.

$$(42) \quad \tilde{s}_\lambda[\Xi_{(k,\mu)}] = \sum_{\nu: \lambda // \nu \in B_k} (-1)^{ht(\lambda // \nu)} \tilde{s}_\nu[\Xi_\mu] = \tilde{s}_\lambda[\Xi_\mu] + \sum_{\nu: \lambda / \nu \in B_k} (-1)^{ht(\lambda / \nu)} \tilde{s}_\nu[\Xi_\mu]$$

Proof. Recall that the Murnaghan-Nakayama rule says that

$$(43) \quad p_k s_\lambda = \sum_{\nu: \lambda \in B_k} (-1)^{ht(\nu / \lambda)} s_\nu$$

and similarly by duality,

$$(44) \quad p_k^\perp s_\lambda = \sum_{\lambda / \nu \in B_k} (-1)^{ht(\lambda / \nu)} s_\nu.$$

where p_k^\perp denotes the adjoint to p_k with respect to the inner product.

Next we calculate by translating the evaluation of $\tilde{s}_\lambda[\Xi_{(k,\mu)}]$ to a symmetric function scalar product.

$$\begin{aligned} \tilde{s}_\lambda[\Xi_{(k,\mu)}] &= \chi^{(|\mu|+k-|\lambda|, \lambda)}(k, \mu) \\ &= \langle s_{(|\mu|+k-|\lambda|, \lambda)}, p_\mu p_k \rangle \\ &= \langle p_k^\perp s_{(|\mu|+k-|\lambda|, \lambda)}, p_\mu \rangle \\ &= \sum_{\nu: \lambda // \nu \in B_k} (-1)^{ht(\lambda // \nu)} \langle s_{(|\mu|-|\nu|, \nu)}, p_\mu \rangle \\ (45) \quad &= \sum_{\nu: \lambda // \nu \in B_k} (-1)^{ht(\lambda // \nu)} \tilde{s}_\nu[\Xi_\mu] \end{aligned}$$

Now to complete the statement of the lemma, we note that a k -border strip that starts in the first row of $(|\mu| + k - |\nu|, \nu)$ lies only in the first row because we assume that $|\mu| > 2|\nu|$. Hence one of the terms where $\lambda // \nu$ is a k border strip is $\nu = \lambda$. All the others partitions such that $\lambda // \nu$ is a k border strip will have the k border strip start in the second row or higher and in this case λ / ν will be a k -border strip. Therefore equation (45) is equal to

$$(46) \quad = \tilde{s}_\lambda[\Xi_\mu] + \sum_{\nu: \lambda / \nu \in B_k} (-1)^{ht(\lambda / \nu)} \tilde{s}_\nu[\Xi_\mu]. \quad \square$$

Say that we wish to compute the expansion of a character. Let λ and ν be partitions and choose $r > \max(|\lambda| + \lambda_1, |\nu| + \nu_1)$. Then $(r - |\lambda|, \lambda)$ and $(r - |\nu|, \nu)$ are partitions and we may compute

$$\begin{aligned} \sum_{\mu \vdash r} \frac{1}{z_\mu} \tilde{s}_\lambda[\Xi_\mu] \tilde{s}_\nu[\Xi_\mu] &= \sum_{\mu \vdash r} \frac{1}{z_\mu} \chi^{(r-|\lambda|, \lambda)}(\mu) \chi^{(r-|\nu|, \nu)}(\mu) \\ (47) \quad &= \frac{1}{r!} \sum_{\sigma \in S_r} \chi^{(r-|\lambda|, \lambda)}(\sigma) \chi^{(r-|\nu|, \nu)}(\sigma) \end{aligned}$$

By the orthogonality of symmetric group characters, this sum is equal to 1 if $\lambda = \nu$ and 0 otherwise.

To apply this principle more generally, and extract a coefficient of the irreducible character basis in a character f , take $r > 2\deg(f)$. Then if $f = \sum_{\lambda} c_{\lambda} \tilde{s}_{\lambda}$, then for all λ in the support of f , $(r - |\lambda|, \lambda)$ will always be a partition. Therefore the coefficient of \tilde{s}_{λ} in f is equal to $\sum_{\mu \vdash r} \frac{1}{z_{\mu}} \tilde{s}_{\lambda}[\Xi_{\mu}] f[\Xi_{\mu}]$. The following is equivalent to Halverson's [Hal] Murnaghan-Nakayama rule for partition algebra characters.

Theorem 19. *For $k > 0$,*

$$(48) \quad p_k \tilde{s}_{\lambda} = \sum_{\nu} \left(\sum_{d|k} \sum_{\alpha} (-1)^{ht(\lambda//\alpha) + ht(\nu//\alpha)} \right) \tilde{s}_{\nu}$$

where the inner sum is over all partitions α such that both $\lambda//\alpha$ and $\nu//\alpha$ are border strips of size d .

Proof. Our proof follows the computation of Halverson [Hal], but in the language of symmetric functions using the irreducible character basis. We can apply this principle to compute the coefficient of \tilde{s}_{ν} in $p_k \tilde{s}_{\lambda}$. To begin, we note that $p_k[\Xi_{\mu}] = \sum_{d|k} dm_d(\mu)$ and choose an r sufficiently large. We compute the coefficient by the expression

$$(49) \quad \sum_{\mu \vdash r} \frac{1}{z_{\mu}} \tilde{s}_{\nu}[\Xi_{\mu}] p_k[\Xi_{\mu}] \tilde{s}_{\lambda}[\Xi_{\mu}] = \sum_{d|k} \sum_{\mu \vdash r} \frac{dm_d(\mu)}{z_{\mu}} \tilde{s}_{\nu}[\Xi_{\mu}] \tilde{s}_{\lambda}[\Xi_{\mu}].$$

Now the non-zero terms of the sum over μ occur when $m_d(\mu) > 0$ and in this case $\frac{dm_d(\mu)}{z_{\mu}} = \frac{1}{z_{\mu-(d)}}$. This is equivalent to summing over all partitions $\bar{\mu}$ of size $r - d$ and $\mu = (d, \bar{\mu})$.

Therefore we have

$$(50) \quad \begin{aligned} \sum_{\mu \vdash r} \frac{1}{z_{\mu}} \tilde{s}_{\nu}[\Xi_{\mu}] p_k[\Xi_{\mu}] \tilde{s}_{\lambda}[\Xi_{\mu}] &= \sum_{d|k} \sum_{\bar{\mu} \vdash r-d} \frac{1}{z_{\bar{\mu}}} \tilde{s}_{\nu}[\Xi_{(d, \bar{\mu})}] \tilde{s}_{\lambda}[\Xi_{(d, \bar{\mu})}] \\ &= \sum_{d|k} \sum_{\bar{\mu} \vdash r-d} \frac{1}{z_{\bar{\mu}}} \sum_{\alpha} \sum_{\beta} (-1)^{ht(\lambda//\alpha) + ht(\nu//\beta)} \tilde{s}_{\alpha}[\Xi_{\bar{\mu}}] \tilde{s}_{\beta}[\Xi_{\bar{\mu}}] \\ &= \sum_{d|k} \sum_{\alpha} \sum_{\beta} (-1)^{ht(\lambda//\alpha) + ht(\nu//\beta)} \sum_{\bar{\mu} \vdash r-d} \frac{1}{z_{\bar{\mu}}} \tilde{s}_{\alpha}[\Xi_{\bar{\mu}}] \tilde{s}_{\beta}[\Xi_{\bar{\mu}}] \\ &= \sum_{d|k} \sum_{\alpha} (-1)^{ht(\lambda//\alpha) + ht(\nu//\alpha)} \end{aligned}$$

where the sum over α is such that $\lambda//\alpha$ is a border strip of size d and the sum over β is $\nu//\beta$ is a border strip of size d . The last equality holds by the orthogonality relations on the symmetric group characters. \square

An expansion of Theorem 19 using the second equality in Lemma 18 yields the following alternate expression for the Murnaghan-Nakayama rule for the irreducible character basis.

Corollary 20. *For $k > 0$,*

$$\begin{aligned} p_k \tilde{s}_\lambda = & \text{divisors}(k) \tilde{s}_\lambda + \sum_{d|k} \sum_{\nu} \sum_{\substack{\alpha: \nu/\alpha \in B_d \\ \lambda/\alpha \in B_d}} (-1)^{ht(\lambda/\alpha) + ht(\nu/\alpha)} \tilde{s}_\nu \\ & + \sum_{d|k} \sum_{\nu: \lambda/\nu \in B_d} (-1)^{ht(\lambda/\nu)} \tilde{s}_\nu + \sum_{d|k} \sum_{\nu: \nu/\lambda \in B_d} (-1)^{ht(\nu/\lambda)} \tilde{s}_\nu \end{aligned}$$

where $\text{divisors}(k)$ is equal to the number of divisors of k .

7. PRODUCTS OF INDUCED TRIVIAL CHARACTERS

A combinatorial interpretation for the Kronecker product of two complete symmetric functions expanded in the complete basis is listed as Exercise 23 (e) in section I.7 of [Mac] and Exercise 7.84 (b) in [Stanley]. The earliest reference to this result that we know of is due to Garsia and Remmel [GR]. As we are interested in the stable case, we state the version where

$$(51) \quad h_{(n-|\lambda|, \lambda)} * h_{(n-|\mu|, \mu)} = \sum_M \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\mu)} h_{M_{ij}}$$

summed over all matrices M of non-negative integers with $\ell(\lambda) + 1$ rows, $\ell(\mu) + 1$ columns and row sums are given by the vectors $(n - |\lambda|, \lambda)$ and column sums by the vector $(n - |\mu|, \mu)$. By equation (15) this Kronecker product is also given by the outer product of \tilde{h}_λ and \tilde{h}_μ since the coefficient of \tilde{h}_ν in $\tilde{h}_\lambda \tilde{h}_\mu$ is equal to the coefficient of $h_{(n-|\nu|, \nu)}$ in $h_{(n-|\lambda|, \lambda)} * h_{(n-|\mu|, \mu)}$ as long as n is sufficiently large (in this case, greater than or equal to $|\mu| + |\lambda|$).

Let S be a multiset and T a set. The restriction of S to T is the multiset $S|_T = \{\{v \in S : v \in T\}\}$. We can define the restriction of a multiset partition to the content T by $\pi|_T = \{\{S|_T : S \in \pi\}\}$. If necessary in this operation we throw away empty multisets in $\pi|_T$.

We will use the notation $\pi \# \tau$ to represent a set of multiset partitions that will appear in the product. Let π and τ be multiset partitions on disjoint sets S and T .

$$(52) \quad \pi \# \tau = \{\theta : \theta \vdash S \cup T, \theta|_S = \pi, \theta|_T = \tau\}$$

That is, $\theta \in \pi \# \tau$ means that θ is of the form

$$(53) \quad \theta = \{\{S_{i_1}, \dots, S_{i_{\ell(\pi)-k}}, T_{j_1}, \dots, T_{j_{\ell(\tau)-k}}, S_{i'_1} \cup T_{j'_1}, \dots, S_{i'_k} \cup T_{j'_k}\}\}$$

where $\{i_1, i_2, \dots, i_{\ell(\pi)-k}, i'_1, i'_2, \dots, i'_k\} = \{1, 2, \dots, \ell(\pi)\}$ and $\{j_1, j_2, \dots, j_{\ell(\tau)-k}, j'_1, j'_2, \dots, j'_k\} = \{1, 2, \dots, \ell(\tau)\}$.

We propose the following (different), but equivalent combinatorial interpretation for this product of the induced trivial character basis.

Proposition 21. *For multiset partitions $\pi \vdash S$ and $\theta \vdash T$ where the multisets S and T are disjoint,*

$$\tilde{h}_{\tilde{m}(\pi)} \tilde{h}_{\tilde{m}(\tau)} = \sum_{\theta \in \pi \# \tau} \tilde{h}_{\tilde{m}(\theta)} .$$

Before we show a proof of this proposition by showing an equivalence with Equation (51), we provide an example to try to clarify any subtleties of the notation.

Example 22. Let $\pi = \{\{1\}, \{1\}, \{2\}\}$ and $\tau = \{\{3\}, \{3\}, \{4\}\}$

Below we list the multiset partitions in $\pi \# \tau$ along with the corresponding partition $\tilde{m}(\theta)$.

$$\begin{aligned} \{\{1\}, \{1\}, \{2\}, \{3\}, \{3\}, \{4\}\} &\rightarrow 2211 & \{\{1,3\}, \{1\}, \{2\}, \{3\}, \{4\}\} &\rightarrow 11111 \\ \{\{1\}, \{1\}, \{2,3\}, \{3\}, \{4\}\} &\rightarrow 2111 & \{\{1\}, \{1,4\}, \{2\}, \{3\}, \{3\}\} &\rightarrow 2111 \\ \{\{1\}, \{1\}, \{2,4\}, \{3\}, \{3\}\} &\rightarrow 221 & \{\{1,3\}, \{1,3\}, \{2\}, \{4\}\} &\rightarrow 211 \\ \{\{1,3\}, \{1,4\}, \{2\}, \{3\}\} &\rightarrow 1111 & \{\{1\}, \{1,3\}, \{2,3\}, \{4\}\} &\rightarrow 1111 \\ \{\{1\}, \{1,4\}, \{2,3\}, \{3\}\} &\rightarrow 1111 & \{\{1\}, \{1,3\}, \{2,4\}, \{3\}\} &\rightarrow 1111 \\ \{\{1,3\}, \{1,3\}, \{2,4\}\} &\rightarrow 21 & \{\{1,3\}, \{1,4\}, \{2,3\}\} &\rightarrow 111 \end{aligned}$$

As a consequence of Proposition 21 we conclude

$$(54) \quad \tilde{h}_{21} \tilde{h}_{21} = \tilde{h}_{111} + 4\tilde{h}_{1111} + \tilde{h}_{11111} + \tilde{h}_{21} + \tilde{h}_{211} + 2\tilde{h}_{2111} + \tilde{h}_{221} + \tilde{h}_{2211}$$

or in terms of Kronecker products with $n = 8$,

$$(55) \quad h_{521} * h_{521} = h_{5111} + 4h_{41111} + h_{311111} + h_{521} + h_{4211} + 2h_{32111} + h_{3221} + h_{22211} .$$

Proof. We define a bijection between matrices whose row sums are $(n - |\lambda|, \lambda)$ and whose column sums are $(n - |\mu|, \mu)$ and elements of $\pi \# \tau$ where π is a multiset partition such that $\tilde{m}(\pi) = \lambda$ and τ is a multiset partition such that $\tilde{m}(\tau) = \mu$.

Let M be such a matrix. The first row of this matrix has sum equal to $n - |\lambda|$ and the sum of row i of this matrix represents the number of times that some multiset A repeats in π (it doesn't matter what that multiset is, just that it repeats $\sum_j M_{ij}$ times). The sum of column j of this matrix (for $j > 1$) represents the number of times that a particular part of the multiset B repeats in τ (again, it doesn't matter the content of that multiset, just that it is different than the others). Therefore the entry M_{ij} is the number of times that $A \cup B$ repeats in the multiset $\theta \in \pi \# \tau$. The value of M_{i1} is equal to the number of times that A appears in θ and the value of M_{1j} is the number of times that B appears in θ . \square

Example 23. To ensure that the bijection described in the proof is clear we show the correspondence between some specific multiset partitions and the non-negative integer matrices to which they correspond. The second and third row will represent the multiplicities of $\{1\}$ and $\{2\}$ respectively. The second and third column will represent the multiplicities of $\{3\}$ and $\{4\}$ respectively. Rather than consider all multiset partitions, we will consider only the 4 that we calculated in the last example that have $\tilde{m}(\pi) = 1111$.

$$\begin{aligned}
\{\{1, 3\}, \{1, 4\}, \{2\}, \{3\}\} &\leftrightarrow \begin{bmatrix} n-4 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\
\{\{1\}, \{1, 3\}, \{2, 3\}, \{4\}\} &\leftrightarrow \begin{bmatrix} n-4 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
\{\{1\}, \{1, 4\}, \{2, 3\}, \{3\}\} &\leftrightarrow \begin{bmatrix} n-4 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \\
\{\{1\}, \{1, 3\}, \{2, 4\}, \{3\}\} &\leftrightarrow \begin{bmatrix} n-4 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

The multiset partition notation is therefore not significantly different than the integer matrices notation, but the are distinct advantages to an interpretation in terms of multiset partitions. The main one is that the notion of multisets in the context of symmetric functions leads to the combinatorial objects of multiset tableaux that can be used as a possible object to keep track of stable Kronecker coefficients.

8. APPENDIX I: EVALUATIONS OF SYMMETRIC FUNCTIONS AT ROOTS OF UNITY

We refer readers to [Lascoux, Lemma 5.10.1] for the following result which we will use as our starting point for evaluations of symmetric functions. In the following expressions, the notation $r|n$ indicates shorthand for “ r divides n .”

Proposition 24. *For $r \geq 0$, $h_0[\Xi_r] = e_0[\Xi_r] = p_0[\Xi_r] = 1$. In addition, for $n > 0$,*

$$(56) \quad h_n[\Xi_r] = \delta_{r|n}, \quad p_n[\Xi_r] = r\delta_{r|n}, \quad e_n[\Xi_r] = (-1)^{r-1}\delta_{r=n}.$$

We will need to take this further and give a combinatorial interpretation for the evaluation of $h_\lambda[\Xi_\mu]$ and $e_\lambda[\Xi_\mu]$ in order to make a connection with character symmetric functions.

8.1. Complete symmetric functions evaluated at roots of unity.

Proposition 25. *For a nonnegative integer n and a partition μ , $h_n[\Xi_\mu]$ is equal to the number of weak compositions α of size n and length $\ell(\mu)$ such that μ_i divides α_i .*

Proof. The alphabet addition formula for h_n says

$$(57) \quad h_n[X_1, X_2, \dots, X_r] = \sum_{\substack{\alpha \models_w n \\ \ell(\alpha)=r}} h_{\alpha_1}[X_1] h_{\alpha_2}[X_2] \cdots h_{\alpha_r}[X_r]$$

therefore evaluating at Ξ_μ , we have

$$(58) \quad h_n[\Xi_\mu] = \sum_{\substack{\alpha \models_w n \\ \ell(\alpha)=\ell(\mu)}} \prod_{i=1}^{\ell(\mu)} h_{\alpha_i}[\Xi_{\mu_i}]$$

where the sum is over all weak compositions of n (that is, 0 parts are allowed) such that the length of the composition (including the 0 parts) is equal to the length of the partition μ . Since $h_{\alpha_i}[\Xi_{\mu_i}] = 0$ unless $\mu_i \mid \alpha_i$, the product contributes 1 to the sum if and only if the sum of the α_i is n and μ_i divides α_i . Therefore this expression represents the number of weak compositions of size n and length $\ell(\mu)$ such that $\mu_i \mid \alpha_i$. \square

Example 26. To evaluate $h_2[\Xi_\mu]$, we can reduce the computation by expressing it in terms of $h_n[\Xi_r]$.

$$(59) \quad h_2[\Xi_\mu] = \sum_{i=1}^{\ell(\mu)} h_2[\Xi_{\mu_i}] + \sum_{1 \leq i < j \leq \ell(\mu)} h_1[\Xi_{\mu_i}] h_1[\Xi_{\mu_j}]$$

Now we know from Proposition 24 that $h_2[\Xi_{\mu_i}] = 1$ if and only if $\mu_i = 1$ or 2, hence

$$(60) \quad \sum_{i=1}^{\ell(\mu)} h_2[\Xi_{\mu_i}] = m_1(\mu) + m_2(\mu)$$

In addition, Proposition 24 implies $h_1[\Xi_{\mu_i}] = 1$ if and only if $\mu_i = 1$ hence

$$(61) \quad \sum_{1 \leq i < j \leq \ell(\mu)} h_1[\Xi_{\mu_i}] h_1[\Xi_{\mu_j}] = \binom{m_1(\mu)}{2}.$$

Alternatively, we can see this evaluation in terms of its computation of the combinatorial interpretation. We have that this is the number of weak compositions of length $\ell(\mu)$ of the form $(0^{i-1}, 2, 0^{\ell(\mu)-i})$ where the 2 is in the position i where $\mu_i = 1$ or 2 or of the form $(0^{i-1}, 1, 0^{j-i-1}, 1, 0^{\ell(\mu)-j})$ where $\mu_i = \mu_j = 1$. Clearly there are $m_1(\mu) + m_2(\mu)$ of the first type and $\binom{m_1(\mu)}{2}$ of the second.

Equation (58) allows us to give a combinatorial interpretation for $h_\lambda[\Xi_\mu]$ in terms of sequences of compositions.

Definition 27. Define the set $\mathcal{C}_{\lambda, \mu}$ to be the sequences $\alpha^{(*)} = (\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(\ell(\lambda))})$ of weak compositions $\alpha^{(i)} \models_w \lambda_i$ such that $\ell(\alpha^{(i)}) = \ell(\mu)$ and μ_j divides $\alpha_j^{(i)}$ for each $1 \leq i \leq \ell(\lambda)$.

Example 28. To compute $\mathcal{C}_{(3,1), (3,3,2,2,1)}$ we count pairs (α, β) where α is a weak composition of 3 of length 5 and β is a weak composition of 1 of length 5 such that μ_i divides α_i and β_i where $\mu = (3, 3, 2, 2, 1)$. There are 5 ways of doing this given by the following pairs of compositions

$$((00003), (00001)), ((00021), (00001)), ((00201), (00001)) \\ ((03000), (00001)), ((30000), (00001))$$

Proposition 29. For partitions λ and μ ,

$$(62) \quad h_\lambda[\Xi_\mu] = |\mathcal{C}_{\lambda, \mu}|$$

Proof. Since $h_r[\Xi_\mu]$ is equal to the number of weak compositions α of length $\ell(\mu)$ and size r such that $\mu_j|\alpha_j$ (that is, it is equal to $|\mathcal{C}_{(r),\mu}|$) hence, by the multiplication principle, the expression

$$(63) \quad h_\lambda[\Xi_\mu] = h_{\lambda_1}[\Xi_\mu] h_{\lambda_2}[\Xi_\mu] \cdots h_{\lambda_{\ell(\lambda)}}[\Xi_\mu],$$

is equal to the number of sequences of compositions $\alpha^{(*)} = (\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(\ell(\lambda))})$ such that $\alpha^{(i)}$ is a weak composition of λ_i and of length $\ell(\mu)$ with μ_j divides $\alpha_j^{(i)}$ for each $1 \leq j \leq \ell(\mu)$. \square

This combinatorial interpretation for $h_\lambda[\Xi_\mu]$ is only the starting point. We will give an expression for this quantity in terms of symmetric function coefficients.

Consider the following scalar product which represents the coefficient of p_μ/z_μ in the symmetric function $h_\lambda h_{|\mu|-|\lambda|}$. We define $H_{\lambda,\mu} := \langle h_\lambda h_{|\mu|-|\lambda|}, p_\mu \rangle$. Now since $H_{\lambda,\mu}$ is equal to the coefficient of $\frac{p_\mu}{z_\mu}$, it is also equal to z_μ times the coefficient of p_μ in the symmetric function $h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_{\ell(\lambda)}} h_{|\mu|-|\lambda|}$. Since $h_{\lambda_i} = \sum_{\gamma^{(i)} \vdash \lambda_i} \frac{p_{\gamma^{(i)}}}{z_{\gamma^{(i)}}}$ then one way that we can express this is

$$(64) \quad H_{\lambda,\mu} = \sum_{\gamma^{(*)}} \frac{z_\mu}{z_{\gamma^{(1)}} z_{\gamma^{(2)}} \cdots z_{\mu \setminus \bigcup_i \gamma^{(i)}}}$$

where the sum is over all sequences of partitions $\gamma^{(*)} = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(\ell(\lambda))})$ with $\gamma^{(i)}$ a partition of λ_i for $1 \leq i \leq \ell(\lambda)$ and such that the parts of $\bigcup_i \gamma^{(i)} = \gamma^{(1)} \cup \gamma^{(2)} \cup \dots \cup \gamma^{(\ell(\lambda))}$ are a subset of the parts of μ . The last partition that appears in this denominator is $\mu \setminus \bigcup_i \gamma^{(i)}$ and this is a partition of size $|\mu| - |\lambda|$ of all the parts of μ which are not used in $\gamma^{(1)} \cup \gamma^{(2)} \cup \dots \cup \gamma^{(\ell(\lambda))}$. We are using the convention that $h_{-r} = 0$ for $r > 0$, hence $H_{\lambda,\mu} = 0$ if $|\lambda| > |\mu|$ since we will have $h_{|\mu|-|\lambda|} = 0$.

Now recall that $z_\mu = \prod_{i=1}^{\mu_1} i^{m_i(\mu)} m_i(\mu)!$. This implies that

$$(65) \quad \begin{aligned} \frac{z_\mu}{z_{\gamma^{(1)}} z_{\gamma^{(2)}} \cdots z_{\gamma^{(\ell(\lambda))}} z_{\mu \setminus \bigcup_i \gamma^{(i)}}} &= \prod_{i=1}^{\mu_1} \frac{m_i(\mu)!}{m_i(\gamma^{(1)})! m_i(\gamma^{(2)})! \cdots m_i(\gamma^{(\ell(\lambda))})! m_i(\mu \setminus \bigcup_i \gamma^{(i)})!} \\ &= \prod_{i=1}^{\mu_1} \binom{m_i(\mu)}{m_i(\gamma^{(1)}), m_i(\gamma^{(2)}), \dots, m_i(\gamma^{(\ell(\lambda))})} \end{aligned}$$

We can state this as the following combinatorial result.

Proposition 30. *For partitions λ and μ , $H_{\lambda,\mu}$ is equal to the number of ways that some of the cells of the diagram of μ can be filled with the labels $\{1, 2, \dots, \ell(\lambda)\}$ such that the whole row is given the same label and in total λ_j cells are labeled with the integer j for $1 \leq j \leq \ell(\lambda)$.*

Proof. We have established by Equations (64) and (65) that

$$(66) \quad H_{\lambda,\mu} = \langle h_{|\mu|-|\lambda|} h_\lambda, p_\mu \rangle = \sum_{\gamma^{(*)}} \prod_{i=1}^{\mu_1} \binom{m_i(\mu)}{m_i(\gamma^{(1)}), m_i(\gamma^{(2)}), \dots, m_i(\gamma^{(\ell(\lambda))})}$$

where the sum is over all sequences of partitions $\gamma^{(*)} = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(\ell(\lambda))})$ with $\gamma^{(j)}$ a partition of λ_j for $1 \leq j \leq \ell(\lambda)$.

We will show that the fillings of the partition with labels described in the statement of the proposition are in bijection with sequences of partitions and sequences of positions counted by the multinomial coefficients in the expression.

Take a filling of some of the rows of the partition μ such that the whole row is assigned the same label and the total number of cells with label j is equal to λ_j . The rows of μ that are labeled with j determine a partition $\gamma^{(j)}$ of size λ_j . This determines the sequence $\gamma^{(*)}$.

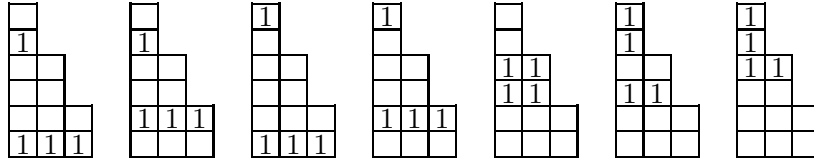
Now among the $m_i(\mu)$ parts of the partition μ that are of size i , $m_i(\gamma^{(j)})$ parts are labeled with label j . There are precisely

$$(67) \quad \binom{m_i(\mu)}{m_i(\gamma^{(1)}), m_i(\gamma^{(2)}), \dots, m_i(\gamma^{(\ell(\lambda))})}$$

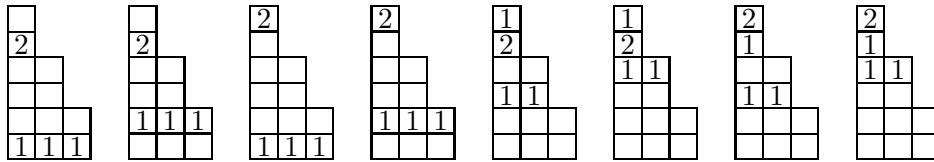
ways this can be done. In other words, there is a sequence of positions which determines where the $m_i(\gamma^{(j)})$ parts of size i are chosen from the $m_i(\mu)$ parts of μ of size i .

The reverse bijection is found by taking a sequence of partitions $\gamma^{(*)}$ and sequence of positions which tells us which of the $m_i(\mu)$ parts of size i of the partition which are filled with the labels j . \square

Example 31. To calculate $H_{(4),(332211)}$ we need to label some of the rows of the partition with just one label such that the number of cells labeled is equal to 4. The following diagrams are the only ways that this can be done.



Therefore $H_{(4),(332211)} = 7$. Similarly, to compute $H_{(31),(332211)}$ we fill the rows of diagram of the partition $(3, 3, 2, 2, 1, 1)$ with labels of three 1's and a 2 such that the whole row is given the same label.



Therefore $H_{(31),(332211)} = 8$.

Recall that we indicate that π is a multiset partition of the multiset with the letter i occurring λ_i times ($1 \leq i \leq \ell(\lambda)$) by the symbol $\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}$. Recall also that $\tilde{m}(\pi)$ is the integer partition of size equal to the number of parts in π where the parts of $\tilde{m}(\pi)$ are the multiplicities of the multisets which appear in π .

A multiset partition of a multiset does not naturally have order on the parts but we will need one for establishing a unique mapping. The order is not especially important, but we

need it to be consistent with how we label column strict tableaux so we choose to put the sets which occur most often first, and among those that occur the same number of times we use lexicographic order if the elements of the set are read in increasing order.

Example 32. In the multiset $\{1^{12}, 2^7, 3^2\}$ we have an example multiset partition

$$(68) \quad \pi = \{\{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 1, 1\}, \{1, 1, 1\}, \{1, 2, 2, 3\}, \{1, 2, 2, 3\}, \{1\}\}$$

The parts $\{1, 2\}$ occur first because they occur 3 times, $\{1, 1, 1\}$ and $\{1, 2, 2, 3\}$ each occur twice so they are each second and $\{1, 1, 1\}$ is before $\{1, 2, 2, 3\}$ because $111 <_{lex} 1223$. Finally $\{1\}$ only occurs once, hence it is last.

Definition 33. Let $\mathcal{P}_{\lambda, \mu}$ be the set of pairs (π, T) where π is a multiset partition of the multiset $\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}$ and T is a filling of some of the cells of the diagram of the partition μ with content $\gamma = \tilde{m}(\pi)$ and all cells in the same row have the same label.

A restatement of Proposition 30 would be that

$$(69) \quad \sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}} H_{\tilde{m}(\pi), \mu} = |\mathcal{P}_{\lambda, \mu}|.$$

Example 34. The set $\mathcal{P}_{(3,1), (3,3,2,2,1)}$ consists of the following 5 pairs of multiset partitions of the multiset $\{1, 1, 1, 2\}$ and fillings of the diagram for $(3, 3, 2, 2, 1)$.

$$\begin{aligned} & (\{\{1, 1, 1, 2\}\}, \begin{array}{|c|c|c|} \hline 1 & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}), \quad (\{\{1\}, \{1\}, \{1, 2\}\}, \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 1 & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}), \quad (\{\{1\}, \{1\}, \{1, 2\}\}, \begin{array}{|c|c|c|} \hline 2 & & \\ \hline 1 & 1 & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array}), \\ & (\{\{1\}, \{1\}, \{1\}, \{2\}\}, \begin{array}{|c|c|c|} \hline 2 & & \\ \hline & & \\ \hline 1 & 1 & 1 \\ \hline & & \\ \hline & & \\ \hline \end{array}), \quad (\{\{1\}, \{1\}, \{1\}, \{2\}\}, \begin{array}{|c|c|c|} \hline 2 & & \\ \hline & & \\ \hline & & \\ \hline 1 & 1 & 1 \\ \hline & & \\ \hline \end{array}) \end{aligned}$$

Definition 35. Let $\mathcal{T}_{\lambda, \mu}$ be the set of fillings of some of the cells of the partition μ with content λ such that any number of labels can go into the same cell but all cells in the same row must have the same multiset of labels.

Example 36. The set $\mathcal{T}_{(3,1), (3,3,2,2,1)}$ consists of the following 5 fillings

$$\begin{array}{|c|c|c|} \hline w & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 12 & & \\ \hline 1 & 1 & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 12 & & \\ \hline 1 & 1 & \\ \hline & & \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & & \\ \hline & & \\ \hline 1 & 1 & 1 \\ \hline & & \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & & \\ \hline & & \\ \hline & & \\ \hline 1 & 1 & 1 \\ \hline & & \\ \hline \end{array}$$

where the w in the first diagram is $w = 1112$ (w represents the multiset of labels $\{1, 1, 1, 2\}$).

The last two examples suggest the following proposition.

Proposition 37. For partitions λ and μ ,

$$(70) \quad \sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}} H_{\tilde{m}(\pi), \mu} = |\mathcal{T}_{\lambda, \mu}|$$

Proof. By Proposition 30 we have established that

$$(71) \quad \sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}} H_{\tilde{m}(\pi), \mu} = |\mathcal{P}_{\lambda, \mu}|,$$

hence it remains to show that there is a bijection between the elements of $\mathcal{P}_{\lambda, \mu}$ and $\mathcal{T}_{\lambda, \mu}$.

Take a pair $(\pi, T) \in \mathcal{P}_{\lambda, \mu}$. Since the order on the parts of the multiset partition of π puts them in weakly decreasing order dependent on the number of times that they occur, the label 1 in T can be replaced by the first multiset which occurs in π , the 2 can be replaced by the second multiset which occurs, etc. The result will be a filling T' which is an element of $\mathcal{T}_{\lambda, \mu}$ because all rows of T have the same labels (and so will be the case with T') and the content of T' will be the same as the multiset and so will be $\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}$.

Example 38. Consider the pair $(\pi, T) \in \mathcal{P}_{(12,7,2), (3,3,2,2,1)}$ where

$$\pi = \{\{1, 2\}, \{1, 2\}, \{1, 2\}, \{1, 1, 1\}, \{1, 1, 1\}, \{1, 2, 2, 3\}, \{1, 2, 2, 3\}, \{1\}\}$$

and T is the filling

4		
2	2	
3	3	
1	1	1

This is mapped to the filling T' equal to

1		
111	111	
1223	1223	
12	12	12

where $T' \in \mathcal{T}_{(12,7,2), (3,3,2,2,1)}$. The map is to replace the labels of T with the parts of π .

As long as the order in which the multisets occur in π is fixed, the map from pairs $(\pi, T) \in \mathcal{P}_{\lambda, \mu}$ to $T' \in \mathcal{T}_{\lambda, \mu}$ is invertible since we can recover the multiset partition of a multiset π as the set of labels in T' and the filling T is just the filling T' with each of the sets replaced by the integer order in which the set appears in π . \square

We are now prepared to provide a proof of the following result which was first stated as Theorem 2 in Section 3 and used to provide a definition of the induced trivial character basis.

Theorem 39. *For all partitions λ ,*

$$(72) \quad h_\lambda[\Xi_\mu] = \sum_{\pi \Vdash \llbracket 1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell} \rrbracket} H_{\tilde{m}(\pi), \mu}.$$

Proof. We have already established in Proposition 29 that $h_\lambda[\Xi_\mu] = |\mathcal{C}_{\lambda, \mu}|$ and in Proposition 37 that

$$(73) \quad \sum_{\pi \Vdash \llbracket 1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell} \rrbracket} H_{\tilde{m}(\pi), \mu} = |\mathcal{P}_{\lambda, \mu}| = |\mathcal{T}_{\lambda, \mu}|.$$

In order to show this result we will provide a bijection between $\mathcal{C}_{\lambda, \mu}$ and $\mathcal{T}_{\lambda, \mu}$.

We start with a filling $T \in \mathcal{T}_{\lambda, \mu}$ and define a list of weak compositions

$$\alpha^{(*)} = (\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(\ell(\lambda))})$$

where $\alpha_i^{(d)}$ is equal to the number of labels d in the i^{th} row of T . Since the content of T is equal to the multiset λ , we know that $\alpha^{(d)}$ will be a weak composition of λ_d and because in row i the multiset of labels is the same for each cell in a row μ_i , it must be that μ_i divides $\alpha_i^{(d)}$.

This procedure is reversible since starting with a sequence of weak compositions $\alpha^{(*)} \in \mathcal{C}_{\lambda, \mu}$, we can place $\alpha_i^{(d)}/\mu_i$ labels of d in each cell of the i^{th} row of the diagram μ to recover the filling $T' \in \mathcal{T}_{\lambda, \mu}$.

We conclude that

$$(74) \quad h_\lambda[\Xi_\mu] = |\mathcal{C}_{\lambda, \mu}| = |\mathcal{T}_{\lambda, \mu}| = |\mathcal{P}_{\lambda, \mu}| = \sum_{\pi \Vdash \llbracket 1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell} \rrbracket} H_{\tilde{m}(\pi), \mu}. \quad \square$$

8.2. Elementary symmetric functions evaluated at roots of unity. To extend this further, we determine the evaluation of an elementary symmetric function at Ξ_μ . For a subset $S = \{i_1, i_2, \dots, i_{|S|}\} \subseteq \{1, 2, \dots, \ell(\mu)\}$, let μ_S denote the sub-partition $(\mu_{i_1}, \mu_{i_2}, \dots, \mu_{i_{|S|}})$. This implies (as we had in equation (58)) that

$$(75) \quad e_n[\Xi_\mu] = \sum_{\substack{\alpha \models_w n \\ \ell(\alpha) = \ell(\mu)}} \prod_{i=1}^{\ell(\mu)} e_{\alpha_i}[\Xi_{\mu_i}] = \sum_{S: |\mu_S| = n} \prod_{i \in S} e_{\mu_i}[\Xi_{\mu_i}] = \sum_{S: |\mu_S| = n} (-1)^{n+|S|}$$

where the sum is over all subsets $S \subseteq \{1, 2, \dots, \ell(\mu)\}$ such that $|\mu_S| = n$.

Definition 40. Define the set $\bar{\mathcal{C}}_{\lambda, \mu}$ to be the set of sequences $(S^{(1)}, S^{(2)}, \dots, S^{(\ell(\lambda))})$ where each $S^{(i)}$ is a subset such that $|\mu_{S^{(i)}}| = \lambda_i$.

Since $e_\lambda[\Xi_\mu] = e_{\lambda_1}[\Xi_\mu] e_{\lambda_2}[\Xi_\mu] \cdots e_{\lambda_{\ell(\lambda)}}[\Xi_\mu]$, it implies that we have the following Proposition for evaluating this value.

Proposition 41. *For partitions λ and μ ,*

$$(76) \quad e_\lambda[\Xi_\mu] = \sum_{S^{(*)} \in \bar{\mathcal{C}}_{\lambda, \mu}} (-1)^{|\lambda| + |S^{(*)}|}$$

where $|S^{(*)}| = \sum_{i=1}^{\ell(\lambda)} |S^{(i)}|$.

Example 42. Consider $n = 4$ and then to evaluate $e_4[\Xi_{3211}]$ there are three subsets of parts of $(3, 2, 1, 1)$ which sum to 4, namely, $\{1, 3\}$, $\{1, 4\}$ and $\{2, 3, 4\}$. The first two are counted with weight 1 and the third has weight -1 , hence $e_4[\Xi_{3211}] = 1 + 1 - 1 = 1$.

To evaluate $e_{31}[\Xi_{3211}]$ we determine that $\bar{\mathcal{C}}_{31,3211} = \{(\{1\}, \{3\}), (\{1\}, \{4\}), (\{2, 3\}, \{3\}), (\{2, 3\}, \{4\}), (\{2, 4\}, \{3\}), (\{2, 4\}, \{4\})\}$. The first two of these have weight $(-1)^{|\lambda|+|S^{(*)}|}$ both equal to 1 and the last four have weight -1 hence $e_{31}[\Xi_{3211}] = -2$.

Now, in addition, we will need to evaluate $HE_{(\lambda|\tau), \mu} := \langle p_\mu, h_{|\mu|-|\lambda|-|\tau|} h_\lambda e_\tau \rangle$ where λ, τ and μ are partitions. By equation (65), we have that this is also equal to

$$HE_{(\lambda|\tau), \mu} = \sum_{\gamma^{(*)}, \nu^{(*)}} \text{sgn}(\nu^{(*)}) \prod_{i=1}^{\mu_1} \binom{m_i(\mu)}{m_i(\gamma^{(1)}), \dots, m_i(\gamma^{(\ell(\lambda))}), m_i(\nu^{(1)}), \dots, m_i(\nu^{(\ell(\tau))})}$$

where the sum is over all sequences of partitions $\gamma^{(*)} = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(\ell(\lambda))})$ where $\gamma^{(j)} \vdash \lambda_j$ and $\nu^{(*)} = (\nu^{(1)}, \nu^{(2)}, \dots, \nu^{(\ell(\tau))})$ where $\nu^{(j)} \vdash \tau_j$ and

$$(77) \quad \text{sgn}(\nu^{(*)}) = (-1)^{\sum_i |\nu^{(i)}| + \ell(\nu^{(i)})}.$$

Note that we are using the convention that if $\bigcup_i \gamma^{(i)} \cup \bigcup_i \nu^{(i)}$ is not a subset of the parts of μ then the weight

$$\prod_{i=1}^{\mu_1} \binom{m_i(\mu)}{m_i(\gamma^{(1)}), \dots, m_i(\gamma^{(\ell(\lambda))}), m_i(\nu^{(1)}), \dots, m_i(\nu^{(\ell(\tau))})}$$

is equal to 0.

Proposition 43. For partitions λ, τ and μ , let $\mathcal{F}_{\lambda, \tau}^\mu$ be the fillings of the diagram for the partition μ with λ_i labels i and τ_j labels j' such that all cells in a row are filled with the same label. For $F \in \mathcal{F}_{\lambda, \tau}^\mu$, the weight of the filling, $\text{wt}(F)$ is equal to -1 raised to the number of cells filled with primed labels plus the number of rows occupied by the primed labels. Then

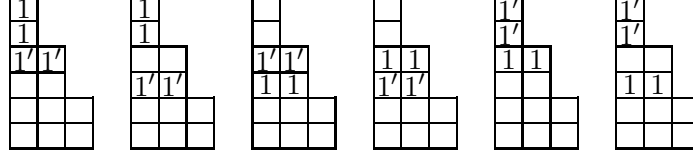
$$(78) \quad HE_{(\lambda|\tau), \mu} = \sum_{F \in \mathcal{F}_{\lambda, \tau}^\mu} \text{wt}(F).$$

Proof. This is precisely the analogous statement to Proposition 30. It follows because if we fix the sequences of partitions $\gamma^{(*)}$ and $\nu^{(*)}$ such that $\gamma^{(i)} \vdash \lambda_i$ and $\nu^{(j)} \vdash \tau_j$, then the quantity

$$(79) \quad \prod_{i=1}^{\mu_1} \binom{m_i(\mu)}{m_i(\gamma^{(1)}), \dots, m_i(\gamma^{(\ell(\lambda))}), m_i(\nu^{(1)}), \dots, m_i(\nu^{(\ell(\tau))})}$$

is precisely the number of F in $\mathcal{F}_{\lambda, \tau}^\mu$ with the rows filled according to the sequences of partitions $\gamma^{(*)}$ and $\nu^{(*)}$. The sign of a filling is constant on this set and is equal to $\text{sgn}(\nu^{(*)})$. \square

Example 44. The following are all the possible fillings of the diagram $(3, 3, 2, 2, 1, 1)$ with two 1's and two 1's such that the rows have the same labels.



Since the weight of the filling is equal to the (-1) raised to the number of cells plus the number of rows occupied by primed entries, the first four have weight -1 and the last two have weight 1 and hence

$$(80) \quad HE_{(2|2), 332211} = -2.$$

We will develop the analogous constructions as were required to prove our formula for the evaluation of h_λ at roots of unity.

We have previously used the notation $\pi \Vdash \{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}$ to indicate that π is a multiset partition of a multiset. We will then use the notation $\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}$ to indicate that π is a *set partition of a multiset*, that is, $\pi = \{P^{(1)}, P^{(2)}, \dots, P^{(\ell(\pi))}\}$ where $P^{(1)} \uplus P^{(2)} \uplus \dots \uplus P^{(\ell(\pi))} = \{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell^{\lambda_\ell}\}$ and each of the $P^{(i)}$ are sets (no repetitions allowed). It is possible that π itself is a multiset since it is possible that $P^{(i)} = P^{(j)}$ when $i \neq j$. In this case we say that π is a set partition of a multiset of content λ (to differentiate from a multiset partition of a multiset).

Now we have previously defined $\tilde{m}(\pi)$ to be a partition representing the multiplicity of the sets that appear in π . Now define $\tilde{m}_e(\pi)$ be a partition representing the multiplicities of the sets with an even number of elements and $\tilde{m}_o(\pi)$ be a partition representing the multiplicities of the sets with an odd number of elements.

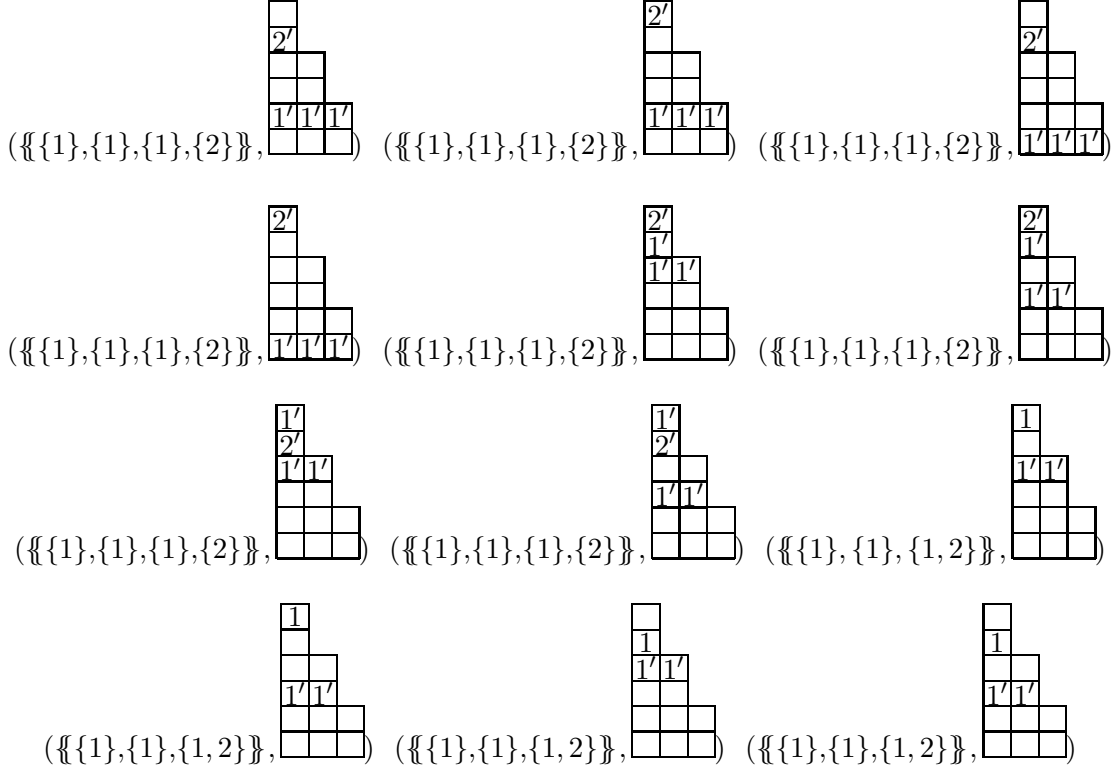
Example 45. Let $\lambda = (5, 3, 3, 2, 1)$ and then

$$(81) \quad \pi = \{\{1, 2, 5\}, \{1, 2\}, \{1, 2\}, \{1, 3\}, \{1, 3\}, \{3, 4\}, \{4\}\}$$

is a set partition of the multiset $\{1^5, 2^3, 3^3, 4^2, 5\}$. The corresponding partition $\tilde{m}(\pi) = (2, 2, 1, 1, 1)$ and $\tilde{m}_e(\pi) = (2, 2, 1)$ and $\tilde{m}_o(\pi) = (1, 1)$. The sequence $\tilde{m}(\pi)$ is a partition of the length of π and $\tilde{m}_e(\pi) \cup \tilde{m}_o(\pi) = \tilde{m}(\pi)$.

Definition 46. For partitions λ and μ , let $\overline{\mathcal{P}}_{\lambda\mu}$ be the set of pairs (π, T) where π is a set partition of the multiset $\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell(\lambda)^{\lambda_{\ell(\lambda)}}\}$. and T is a filling of some of the rows of the diagram for μ with content $\gamma = \tilde{m}_e(\pi)$ consisting of labels $\{1^{\gamma_1}, 2^{\gamma_2}, \dots, \ell(\gamma)^{\gamma_{\ell(\gamma)}}\}$ and some rows filled with content $\tau = \tilde{m}_o(\pi)$ consisting of primed labels $\{1'^{\tau_1}, 2'^{\tau_2}, \dots, \ell(\tau)^{\tau_{\ell(\tau)}}\}$. The weight of a pair (π, T) will be either ± 1 and is equal to -1 raised to the number of prime labels plus the number of rows those labels occupy.

Example 47. Consider the set $\overline{\mathcal{P}}_{(3,1),(3,3,2,2,1,1)}$ that consists of the following 12 pairs of set partitions and fillings



The first four of these pairs have weight $+1$ and the remaining eight have weight -1 .

With these definitions, we can use Proposition 43 to state that

$$(82) \quad \sum_{\pi \vdash \{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell(\lambda)\}} HE_{(\tilde{m}_e(\pi) | \tilde{m}_o(\pi)), \mu} = \sum_{F \in \overline{\mathcal{P}}_{\lambda, \mu}} wt(F) .$$

Next we define a set which is the natural analogue of $\overline{\mathcal{T}}_{\lambda, \mu}$. This set is defined so that it is one step away from the definition of $\overline{\mathcal{P}}_{\lambda, \mu}$ but for each pair (π, T) we replace the entries of T with the sets that make up the parts of π .

Definition 48. For partitions λ and μ let $\overline{\mathcal{T}}_{\lambda, \mu}$ be the fillings of some of the cells of the diagram of the partition μ with subsets of $\{1, 2, \dots, \ell(\lambda)\}$ such that the total content of the filling is $\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell(\lambda)^{\lambda_{\ell(\lambda)}}\}$ and such that all cells in the same row have the same subset of entries. We will define the weight of one of these fillings to be -1 to the power of the size of λ plus the number of rows whose cells are occupied by a set of odd size (this is also equal to the number of cells plus the number of rows occupied by the sets of odd size).

Example 49. The following 12 tableaux are the elements of $\overline{\mathcal{T}}_{(3,1),(3,3,2,2,1,1)}$.

$\begin{array}{ c c c c } \hline & & & \\ \hline 2 & & & \\ \hline & & & \\ \hline & & & \\ \hline 1 & 1 & 1 & \\ \hline & & & \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 2 & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline 1 & 1 & 1 & \\ \hline & & & \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline & & & \\ \hline 2 & & & \\ \hline & & & \\ \hline & & & \\ \hline 1 & 1 & 1 & \\ \hline & & & \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 2 & & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline 1 & 1 & 1 & \\ \hline & & & \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 2 & & & \\ \hline 1 & & & \\ \hline 1 & 1 & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 2 & & & \\ \hline 1 & & & \\ \hline & & & \\ \hline 1 & 1 & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$
$\begin{array}{ c c c c } \hline 1 & & & \\ \hline 2 & & & \\ \hline 1 & 1 & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 1 & & & \\ \hline 2 & & & \\ \hline & & & \\ \hline 1 & 1 & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 12 & & & \\ \hline & & & \\ \hline 1 & 1 & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline 12 & & & \\ \hline & & & \\ \hline & & & \\ \hline 1 & 1 & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline & & & \\ \hline 12 & & & \\ \hline 1 & 1 & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$	$\begin{array}{ c c c c } \hline & & & \\ \hline 12 & & & \\ \hline & & & \\ \hline 1 & 1 & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$

The fillings listed above are in the same order as their isomorphism with the set of pairs $\overline{\mathcal{P}}_{(3,1),(3,3,2,2,1,1)}$ from Example 47. As in that case we have that, the first four of these pairs have weight $+1$ and the remaining eight have weight -1 .

The following result should be clear from the definitions listed above and the examples we have presented.

Lemma 50. *There is a bijection between the sets $\overline{\mathcal{P}}_{\lambda,\mu}$ and $\overline{\mathcal{T}}_{\lambda,\mu}$ and $\overline{\mathcal{C}}_{\lambda,\mu}$ that preserves the weight.*

Corollary 51. *For partitions λ and μ ,*

$$(83) \quad e_{\lambda}[\Xi_{\mu}] = \sum_{\pi \vdash \{\{1^{\lambda_1}, 2^{\lambda_2}, \dots, \ell(\lambda_{\ell})\}\}} \mathcal{H}E_{(\tilde{m}_e(\pi)|\tilde{m}_o(\pi)),\mu}$$

Example 52. There are only three set partitions of $\{\{1^3, 2^2\}\}$. These are

$$\{\{\{1\}, \{1\}, \{1\}, \{2\}, \{2\}\}\}, \{\{\{1\}, \{1\}, \{1, 2\}, \{2\}\}\}, \{\{\{1\}, \{1, 2\}, \{1, 2\}\}\}.$$

Corollary 51 states that

$$(84) \quad e_{32}[\Xi_{\mu}] = \mathcal{H}E_{(\cdot|32),\mu} + \mathcal{H}E_{(1|21),\mu} + \mathcal{H}E_{(2|1),\mu}$$

Remark 53. The expressions $\mathcal{H}E_{(\lambda,\tau),\mu}$ implies that we could define symmetric functions $\tilde{h}e_{(\lambda|\tau)}$ with the property $\tilde{h}e_{(\lambda|\tau)}[\Xi_{\mu}] = \mathcal{H}E_{(\lambda|\tau),\mu} = \langle h_{|\mu|-|\lambda|-|\tau|} h_{\lambda} e_{\tau}, p_{\mu} \rangle$. Some of the results we present in this paper can be generalized to elements $\tilde{h}e_{(\lambda|\tau)}$ which form a spanning set.

9. APPENDIX II: ROOTS OF UNITY AND ZEROS OF MULTIVARIATE POLYNOMIALS

The goal of this section is to prove the following basic proposition.

Proposition 54. *Let $f, g \in \text{Sym}$ be symmetric functions of degree less than or equal to some positive integer n . Assume that*

$$f[\Xi_{\mu}] = g[\Xi_{\mu}]$$

for all partitions μ such that $|\mu| \leq n$, then

$$f = g$$

as elements of Sym .

To make this proposition clear, we will reduce it to the fact that a univariate polynomial of degree n has at most n zeros and hence a univariate polynomial of degree at most n with more than n zeros must be the 0 polynomial.

To begin, we note that $p_r[\Xi_k] = k$ if k divides r and it is equal to 0 otherwise. In general, we can express any partition μ in exponential notation $\mu = (1^{m_1} 2^{m_2} \dots r^{m_r})$ where m_i are the number of parts of size i in μ . Therefore

$$(85) \quad p_k[\Xi_\mu] = \sum_{d|k} dm_d$$

Hence any symmetric function f evaluated at some set of roots of unity is equal to a polynomial in values m_1, m_2, \dots, m_n where

$$(86) \quad f[\Xi_\mu] = f \Big|_{p_k \rightarrow \sum_{d|k} dm_d} = q(m_1, m_2, \dots, m_n) .$$

Moreover, if we know this polynomial $q(m_1, m_2, \dots, m_d)$ we can use Möbius inversion to recover the symmetric function since if $p_k = \sum_{d|k} dm_d$, then $km_k = \sum_{d|k} \mu(k/d)p_d$ where

$$(87) \quad \mu(r) = \begin{cases} (-1)^d & \text{if } r \text{ is a product of } d \text{ distinct primes} \\ 0 & \text{if } r \text{ is not square free} \end{cases} .$$

Therefore, we also have

$$(88) \quad q \left(p_1, \frac{p_2 - p_1}{2}, \frac{p_3 - p_1}{3}, \dots, \frac{1}{n} \sum_{d|n} \mu(n/d)p_d \right) = f .$$

To show that Proposition 54 is true, we will prove that if $h[\Xi_\mu] = 0$ for all partitions $|\mu| \leq n$, then $h = 0$ as a symmetric function where $h = f - g$. We will do this by considering h as $q(x_1, x_2, \dots, x_n)$ where x_r is replaced by $\frac{1}{r} \sum_{d|r} \mu(r/d)p_d$ and that this multivariate polynomial evaluates to 0 for all (m_1, m_2, \dots, m_n) where $\mu = (1^{m_1} 2^{m_2} \dots n^{m_n})$ is a partition with $|\mu| \leq n$. This is a consequence of the next lemma below.

Define the degree of a monomial so that $\deg(x_i) = i$ and hence $\deg(x_1^{a_1} x_2^{a_2} \dots x_k^{a_k}) = a_1 + 2a_2 + 3a_3 + \dots + ka_k$.

Lemma 55. *Let $q(x_1, x_2, \dots, x_n)$ be an element in a multivariate polynomial ring $\mathbb{Q}[x_1, x_2, \dots, x_n]$ with $\deg(q(x_1, x_2, \dots, x_n)) \leq d$ for some d . If $q(m_1, m_2, \dots, m_n) = 0$ for all sequences (m_1, m_2, \dots, m_n) with $m_i \geq 0$ and $m_1 + 2m_2 + 3m_3 + \dots + nm_n \leq d$, then $q(x_1, x_2, \dots, x_n) = 0$.*

Proof. We argue by induction on the number of variables n . First we note that if $n = 1$, then if $q(x_1)$ is a polynomial of degree $\leq d$ and $q(0) = q(1) = \dots = q(d) = 0$, then $q(x_1) = 0$ because we know that a univariate polynomial of degree r can have at most r roots.

Let our induction assumption be that, if $q(m_1, m_2, \dots, m_{n-1}) = 0$ for all sequences $(m_1, m_2, \dots, m_{n-1})$ with $m_i \geq 0$ and $m_1 + 2m_2 + 3m_3 + \dots + (n-1)m_{n-1} \leq d$, then $q(x_1, x_2, \dots, x_{n-1}) = 0$.

Now assume that our inductive hypothesis is true and consider a polynomial in n variables,

$$(89) \quad q(x_1, x_2, \dots, x_n) = \sum_{i=0}^r q^{(i)}(x_1, x_2, \dots, x_{n-1}) x_n^i$$

where r is the degree of the polynomial q in the variable x_n and $0 \leq r \leq d/n$ and the coefficient of x_n^i is $q^{(i)}(x_1, x_2, \dots, x_{n-1})$ a multivariate polynomial of degree less than or equal to $d - ni$.

We wish to show that $q(x_1, x_2, \dots, x_n)$ is in fact 0. Assume by smallest counterexample that r is the largest exponent of x_n for which there is a non-zero coefficient, then fix $(m_1, m_2, \dots, m_{n-1})$ such that $m_1 + 2m_2 + \dots + (n-1)m_{n-1} \leq d - rn$. Now

$$(90) \quad q(m_1, m_2, \dots, m_{n-1}, m_n) = 0$$

for each $m_n = 0, 1, 2, \dots, r$, hence $q(m_1, m_2, \dots, m_{n-1}, x_n) = 0$ because it is a polynomial of degree at most r with more than r roots and in particular the coefficient $q^{(r)}(m_1, m_2, \dots, m_{n-1})$ is equal to 0. But now we have that $q^{(r)}(x_1, x_2, \dots, x_{n-1})$ is a polynomial of degree $\leq d - rn$ in $n-1$ variables which vanishes at all $(x_1, x_2, \dots, x_{n-1}) = (m_1, m_2, \dots, m_{n-1})$ with $m_1 + 2m_2 + \dots + (n-1)m_{n-1} \leq d - rn$ and hence $q^{(r)}(x_1, x_2, \dots, x_{n-1}) = 0$ by our induction hypothesis. \square

Hence, Proposition 54 follows as a corollary.

Proof. (of Proposition 54) We will show the equivalent statement that if h is a symmetric function of degree less than or equal to n and $h[\Xi_\mu] = 0$ for all $|\mu| \leq n$, then $h = 0$ (by setting $h = f - g$).

This statement is directly equivalent to Lemma 55 since if we replace p_k in h with $\sum_{d|k} dx_d$ then

$$q(x_1, x_2, \dots, x_n) = h \Big|_{p_k \rightarrow \sum_{d|k} dx_d}$$

is a polynomial of degree at most n that has the property that $q(m_1, m_2, \dots, m_n) = h[\Xi_\mu] = 0$ for all $\mu = (1^{m_1} 2^{m_2} \dots n^{m_n})$ for all $|\mu| \leq n$. By Lemma 55, $q(x_1, x_2, \dots, x_n) = 0$ and hence by Equation (88), $h = 0$. \square

We considered the case in Proposition 54 that $f[\Xi_\mu] = g[\Xi_\mu]$ implies $f = g$ when μ is small, but more often we will know that $f[\Xi_\mu] = g[\Xi_\mu]$ for all μ which are partitions greater than some value n . We can reduce the implication to the previous case in the following proposition.

Corollary 56. *Let $f, g \in \text{Sym}$ be symmetric functions of degree less than or equal to some positive integer n . Assume that*

$$f[\Xi_\mu] = g[\Xi_\mu]$$

for all partitions μ such that $|\mu| \geq n$, then

$$f = g$$

as elements of Sym .

Proof. We reduce the conditions of this corollary to the previous case by considering partitions $\bar{\mu}$ of size less than or equal to n and then let $\mu = (n+1, \bar{\mu})$. Since for all $k \leq n$, $p_k[\Xi_{n+1}] = 0$, then $p_k[\Xi_\mu] = p_k[\Xi_{\bar{\mu}}]$.

Since f (and similarly g) are of degree less than or equal to n , then $f = \sum_{|\lambda| \leq n} c_\lambda p_\lambda$ for some coefficients c_λ then $p_\lambda[\Xi_\mu] = p_\lambda[\Xi_{\bar{\mu}}]$ since $p_\lambda[\Xi_{n+1}] = 0$. Therefore $f[\Xi_\mu] = f[\Xi_{\bar{\mu}}]$ and $g[\Xi_\mu] = g[\Xi_{\bar{\mu}}]$ and hence $f[\Xi_\mu] = g[\Xi_\mu]$. By Proposition 54 we can conclude that $f = g$. \square

10. APPENDIX III: EXAMPLES OF S_n MODULES AND THEIR CHARACTERS

A recent paper by Church and Farb [CF] introduces a notion of representation stability. They describe a number of families of representations whose decomposition into irreducible representations is independent of n , if n is sufficiently large. The bases that we introduce in this paper are likely to be a useful tool in finding expressions for their character.

For this section we will assume that the reader is familiar with some basics of representation theory and the algebraic constructions of the tensor $T^k(V)$, exterior tensor $\bigwedge^k(V)$ and symmetric tensor spaces $S^k(V)$.

Example 57. (The k -fold tensor space and its multilinear part). Let $V_n = \mathcal{L}\{v_1, v_2, \dots, v_n\}$ be a GL_n module, with the action

$$(91) \quad v_i A = \sum_{j=1}^n a_{ij} v_j.$$

Let $T^k(V_n)$ denote the k^{th} tensor of V . Stated in another way $T^k(V_n)$ is a S_n module as well as an GL_n module and if we take A to be a diagonal matrix

$$(92) \quad A = \text{diag}(x_1, x_2, \dots, x_n) = \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{bmatrix},$$

then right action of GL_n on a tensor element is given by

$$(93) \quad (v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}) A = x_{i_1} x_{i_2} \cdots x_{i_k} v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$$

Hence the coefficient of the action of $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$ on the basis element $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_k}$ is the monomial $x_{i_1} x_{i_2} \cdots x_{i_k}$.

If we calculate the trace by summing over all basis elements, then we have

$$(94) \quad \text{char}_{GL_n}^{T^k(V_n)}(A) = \sum_{(i_1, i_2, \dots, i_k) \in [n]^k} x_{i_1} x_{i_2} \cdots x_{i_k} = h_{1^k}(x_1, x_2, \dots, x_n)$$

where $[n]^k = [n] \times \cdots \times [n]$ k times.

Thus, the character of $T^k(V_n)$ as a symmetric function is equal to h_{1^k} (both as a GL_n and as an S_n character). This character expands positively in both the induced trivial character basis (by equation (4)) and the irreducible character basis (Theorem 5) and

those expansions describe precisely how it decomposes into submodules. The proposition implies that

$$(95) \quad h_{1^k} = \sum_{\pi \Vdash \{\{1,2,\dots,k\}\}} \tilde{h}_{\tilde{m}(\pi)} = \sum_{d=1}^k S(k,d) \tilde{h}_{1^d} .$$

where $S(k,d)$ is the Stirling number of the second kind (the number of set partitions of k into d parts). Then from Theorem 5 the multiplicity of the irreducible S_n module is given by the following corollary.

Corollary 58. *For n sufficiently large, the multiplicity of the irreducible indexed by the partition $(n - |\lambda|, \lambda)$ in the module $T^k(V)$ is equal to the number of set valued tableaux T of skew-shape $(m, \lambda)/(\lambda_1)$ for some m , and entries filled with non-empty subsets whose total content is $\{1, 2, \dots, k\}$.*

For example the multiplicity of the irreducible indexed by $(6, 1, 1)$ in $T^3(V_8)$ is equal to 6 because there are 6 set tableaux of shape either $(2, 1, 1)$ or $(1, 1, 1)$ with a blank entry in the first row. Those tableaux are

$$\begin{array}{|c|} \hline 3 \\ \hline 2 \\ \hline \end{array} \begin{array}{|c|} \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 3 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 2 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} \begin{array}{|c|} \hline 3 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 3 \\ \hline 12 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 23 \\ \hline 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 2 \\ \hline 13 \\ \hline \end{array} .$$

The multilinear part of $T^k(V_n)$ is the subspace of $T^k(V_n)$ where all entries of the tensors are different. This is an S_n module and not a GL_n module because the space is not closed under the action of an element of GL_n on a k -fold tensor. The multilinear part of $T^k(V_n)$ is a module of dimension $(n)_k = n(n-1) \cdots (n-k+1)$ and is isomorphic to the S_n module acting on the formal sum of lists of length k with distinct entries in $\{1, 2, \dots, n\}$. We will denote it by $T^k(V_n)_\#$.

The module $T^k(V_n)_\#$ is isomorphic to the induced module $\text{ind} \uparrow_{S_k}^{S_n} T^k(V_k)_\#$. Using the Frobenius image of the character of the module, we know that the value of the character at a permutation of cycle structure μ is equal to the scalar product $\langle h_{n-k} h_{1^k}, p_\mu \rangle$ which is equal to $\tilde{h}_{1^k}[\Xi_\mu]$. We can conclude that the character of the multilinear part as a symmetric function is equal to \tilde{h}_{1^k} .

Example 59. (The k -fold symmetric tensor space and its multilinear part). Consider now $S^k(V_n)$, the k^{th} symmetric tensor of $V_n = \mathcal{L}\{v_1, v_2, \dots, v_n\}$ which can be identified with the commutative polynomials of degree k in n variables. Since a basis of this space is $v_{i_1} v_{i_2} \cdots v_{i_k}$ with $i_1 \leq i_2 \leq \cdots \leq i_k$, we can compute the GL_n character (which will be equal to the $S_n \subseteq GL_n$ character) again by summing over basis elements and for a diagonal matrix $A = \text{diag}(x_1, x_2, \dots, x_n)$, $(v_{i_1} v_{i_2} \cdots v_{i_k})A = x_{i_1} x_{i_2} \cdots x_{i_k} (v_{i_1} v_{i_2} \cdots v_{i_k})$

$$(96) \quad \text{char}_{GL_n}^{S^k(V_n)}(A) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k} = h_k(x_1, x_2, \dots, x_n)$$

Thus, the character of $S^k(V_n)$ as a symmetric function is equal to h_k (both as a GL_n and as an S_n character). Its expansion into S_n irreducible elements is given (again) by Theorem 5. We state the specific case in the following corollary.

As an example, the multiplicity of the occurrence in the irreducible indexed by $(4, 2)$ in $S^4(V_6)$ is equal to the number of the following multiset valued tableaux.

For $k > |\bar{\lambda}| + \lambda_2$, the coefficient of q^k in $s_\lambda \left[\frac{1}{1-q} \right]$ and the coefficient $\tilde{s}_{\bar{\lambda}}$ in h_k are equal (there is an easy bijection between the combinatorial interpretations which we leave as an exercise). It is only for k small that we may have to consider a set of combinatorial objects that might have weight ± 1 to determine the multiplicity of χ^λ in the character for $S^k(V_n)$.

$$(97) \quad S^{\lambda_1}(V_n) \otimes S^{\lambda_2}(V_n) \otimes \cdots \otimes S^{\lambda_{\ell(\lambda)}}(V_n)$$

The analogue of the multilinear part of the k -fold tensor $T^k(V_n)$ where we consider the symmetric tensor $S^k(V_n)$ is the subspace of symmetric tensors with distinct entries. This is also isomorphic to the subspace of polynomials of degree k consisting of the square free monomials. This can also be thought of as the S_n -module of the formal sum of subsets of size k of the integers $\{1, 2, \dots, n\}$. It is a space of dimension $\binom{n}{k}$ and we will denote this space by $S^k(V_n)_\#$

More generally, we have that the multilinear part of the module

$$(98) \quad S^{\lambda_1}(V_n) \otimes S^{\lambda_2}(V_n) \otimes \cdots \otimes S^{\lambda_{\ell(\lambda)}}(V_n)$$

Example 61. (The exterior tensor space). Let $\bigwedge^k(V_n)$ be the k^{th} exterior power of V_n . Since all entries of the k -fold wedge are distinct this example is the same as the ‘multilinear part.’ The vector space $\bigwedge^k(V_n)$ is a GL_n module with basis $v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_k}$

with $i_1 < i_2 < \dots < i_k$. Let $A = \text{diag}(x_1, x_2, \dots, x_n)$ which has an action on a basis element as $(v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k})A = x_{i_1}x_{i_2} \dots x_{i_k}(v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k})$. We may compute the character as equal to

$$\begin{aligned} \text{char}_{\bigwedge^k(V_n)}(A) &= \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} (v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k})A \Big|_{v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_k}} \\ &= \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} x_{i_1}x_{i_2} \dots x_{i_k} = e_k(x_1, x_2, \dots, x_n) \end{aligned}$$

We know that $e_k = \tilde{s}_{1^k} + \tilde{s}_{1^{k-1}}$ so we know that $\bigwedge^k(V_n)$ for $n > k$ decomposes into two irreducible submodules indexed by the partitions $(n - k, 1^k)$ and $(n - k + 1, 1^{k-1})$.

11. APPENDIX IV: SAGE AND THE CHARACTER BASES OF SYMMETRIC FUNCTIONS

Sage is an open source symbolic calculation program based on the computer language Python. A large community of mathematicians participate in its support and add to its functionality [sage, sage-combinat]. In particular, the built-in library for symmetric functions includes a large extensible set of functions which makes it possible to do calculations within the ring following closely the mathematical notation that we use in this paper. The language itself has a learning curve, but the contributions made by the community towards the functionality make that a barrier worth overcoming.

In version 6.10 or later of Sage (released Jan 2016) these bases will be available as methods in the ring of symmetric functions.

We demonstrate examples of some of the definitions and results in this paper by Sage calculations.

```
sage: Sym = SymmetricFunctions(QQ)
sage: Sym
Symmetric Functions over Rational Field
sage: st = Sym.irreducible_symmetric_group_character(); st
Symmetric Functions over Rational Field in the irreducible character
basis
sage: s = Sym.Schur(); s
Symmetric Functions over Rational Field in the Schur basis
sage: h = Sym.complete(); h
sage: Symmetric Functions over Rational Field in the homogeneous basis
sage: ht = Sym.induced_trivial_character(); ht
Symmetric Functions over Rational Field in the induced trivial character
basis
```

We can compare the structure coefficients of the irreducible character basis with the Kronecker product of Schur functions whose first part is sufficiently large.

```
sage: st[2]*st[2]
st[] + st[1] + st[1, 1] + st[1, 1, 1] + 2*st[2] + 2*st[2, 1] + st[2, 2]
+ st[3] + st[3, 1] + st[4]
```

```

sage: s[6,2].kronecker_product(s[6,2])
s[4, 2, 2] + s[4, 3, 1] + s[4, 4] + s[5, 1, 1, 1] + 2*s[5, 2, 1]
+ s[5, 3] + s[6, 1, 1] + 2*s[6, 2] + s[7, 1] + s[8]
sage: ht[2,1]*ht[2,1] # Example 22
ht[1, 1, 1] + 4*ht[1, 1, 1, 1] + ht[1, 1, 1, 1, 1] + ht[2, 1]
+ ht[2, 1, 1] + 2*ht[2, 1, 1, 1] + ht[2, 2, 1] + ht[2, 2, 1, 1]

```

We can express one basis in terms of another. If **b1** and **b2** are bases in Sage, then an expression of the form “**b1**(expression in **b2** basis)” will output the expression in the **b1** basis.

```

sage: ht(h[2,2]) # express h_22 in the ht-basis
ht[1] + 3*ht[1, 1] + ht[1, 1, 1] + ht[2] + 2*ht[2, 1] + ht[2, 2]
sage: st(h[2,1]) # Example 6
4*st[] + 7*st[1] + 3*st[1, 1] + 4*st[2] + st[2, 1] + st[3]

```

Symmetric functions have two methods which correspond to operations that we use in this paper. The first is the evaluation of a symmetric function at the eigenvalues of a permutation matrix where the permutation has cycle structure μ . We represented this operation in the paper as $f[\Xi_\mu]$. In Sage, elements of the symmetric functions have the method `eval_at_permutation_roots` which represents this operation. Comparing to Examples 28, 31, 34 and 36 we compute

```

sage: ht[3,1].eval_at_permutation_roots([3,3,2,2,1])
5
sage: ht[4].eval_at_permutation_roots([3,3,2,2,1,1])
7
sage: ht[3,1].eval_at_permutation_roots([3,3,2,2,1,1])
8

```

In Sage, these can be compared to the following coefficients in the power sum basis.

```

sage: p = Sym.powersum(); p
Symmetric Functions over Rational Field in the powersum basis
sage: h[7,3,1].scalar(p[3,3,2,2,1])
5
sage: h[8,4].scalar(p[3,3,2,2,1,1])
7
sage: h[8,3,1].scalar(p[3,3,2,2,1,1])
8

```

The other operation that we define is one that interprets a symmetric function as a symmetric group character and then maps that character to the Frobenius image (or characteristic map) of the character. That is, for a symmetric function f , the function computes

$$(99) \quad \phi_n(f) = \sum_{\mu \vdash n} f[\Xi_\mu] \frac{p_\mu}{z_\mu}.$$

The elements of the symmetric functions in Sage have the method `character_to_frobenius_image` which represents the map ϕ_n .

We have defined the irreducible character and induced trivial character bases so that they have the property $\phi_n(\tilde{s}_\lambda) = s_{(n-|\lambda|, \lambda)}$ and $\phi_n(\tilde{h}_\lambda) = h_{n-|\lambda|} h_\lambda$ respectively if $n \geq |\lambda| + \lambda_1$. If $n < |\lambda| + \lambda_1$, then the corresponding symmetric function will be indexed by a composition.

```
sage: s(st[3,2].character_to_frobenius_image(8))
s[3,3,2]
sage: h(ht[3,2].character_to_frobenius_image(6))
h[3,2,1]
```

It is this operation that is the origin of our definitions since, at least in the beginning of our investigations, the induced trivial character and irreducible character bases for us were the preimages of the Schur and complete symmetric functions in the ϕ_n map.

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